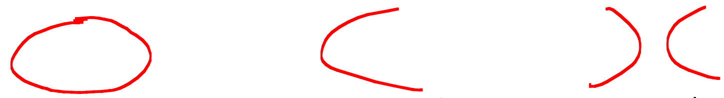
Clemens, A scrapbook of complex curve theory

Chapter 1. Conics / Quadratic Eqt.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

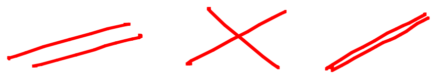
[Cubic (ch2, 3) ; Quartic & Quintic (ch5)]

$/\mathbb{R}$ smooth :



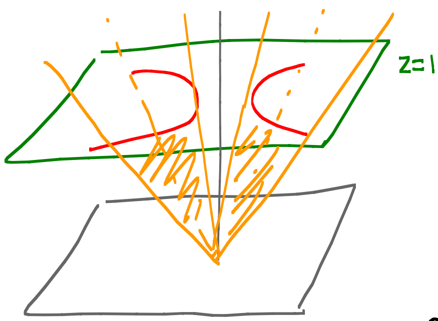
ellipse parabola hyperbola

(other possibilities:



$\cdot \quad \phi \quad \mathbb{R}^2$)

"Cone it" : $\{Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0\} \subseteq \mathbb{R}^3$



Say $xy = 1$ on $\mathbb{R}^2$

becomes $xy = z^2$ on $\mathbb{R}^3$

restrict to $\begin{cases} z = 1 \rightsquigarrow xy = 1 & \text{hyperbola} \\ x = 2 \rightsquigarrow z^2 = 2y & \text{parbola} \\ x+y=4 \rightsquigarrow z^2 + \frac{1}{2}y^2 = 4 & \text{ellipse} \end{cases}$

$\Rightarrow$ unification.

eqt. $\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

change coord. $/\mathbb{R}$ $\Rightarrow \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ s.t. $\varepsilon_i = \pm 1$ or 0, $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_3$, $\varepsilon_2 \geqslant 0$

change coord. $/\mathbb{C}$ $\Rightarrow$ $\varepsilon_i = 0$ or 1

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \to$ usual cone $\quad \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \to$ doubleplane

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \to \quad \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \to \mathbb{C}^3$

- $\{\text{Conics in } \mathbb{CP}^2\} \leftrightarrow \{(A,B,C,D,E,F)\}/\mathbb{C}^\times = \mathbb{CP}^5$

  thru. a point $\Rightarrow$ linear constraint $\rightsquigarrow$ $\overset{\cup\!\mathsf{I}}{\mathbb{CP}^4}$

  Eg. $\exists!$ conic thru. 5 (general) pt. in $\mathbb{CP}^2$.

Similarly, $\exists$ $\mathbb{P}^1$-family (linear) cubic thru. 8 pt. in $\mathbb{CP}^2$.

$\mathbb{CP}^2 \underset{\text{cubic}}{\supset} E_1, E_2, E_3 \ni \{p_1, \ldots, p_8\}$

$E_i = \{f_i(x,y,z) = 0\}$ $\qquad \deg f_i = 3.$

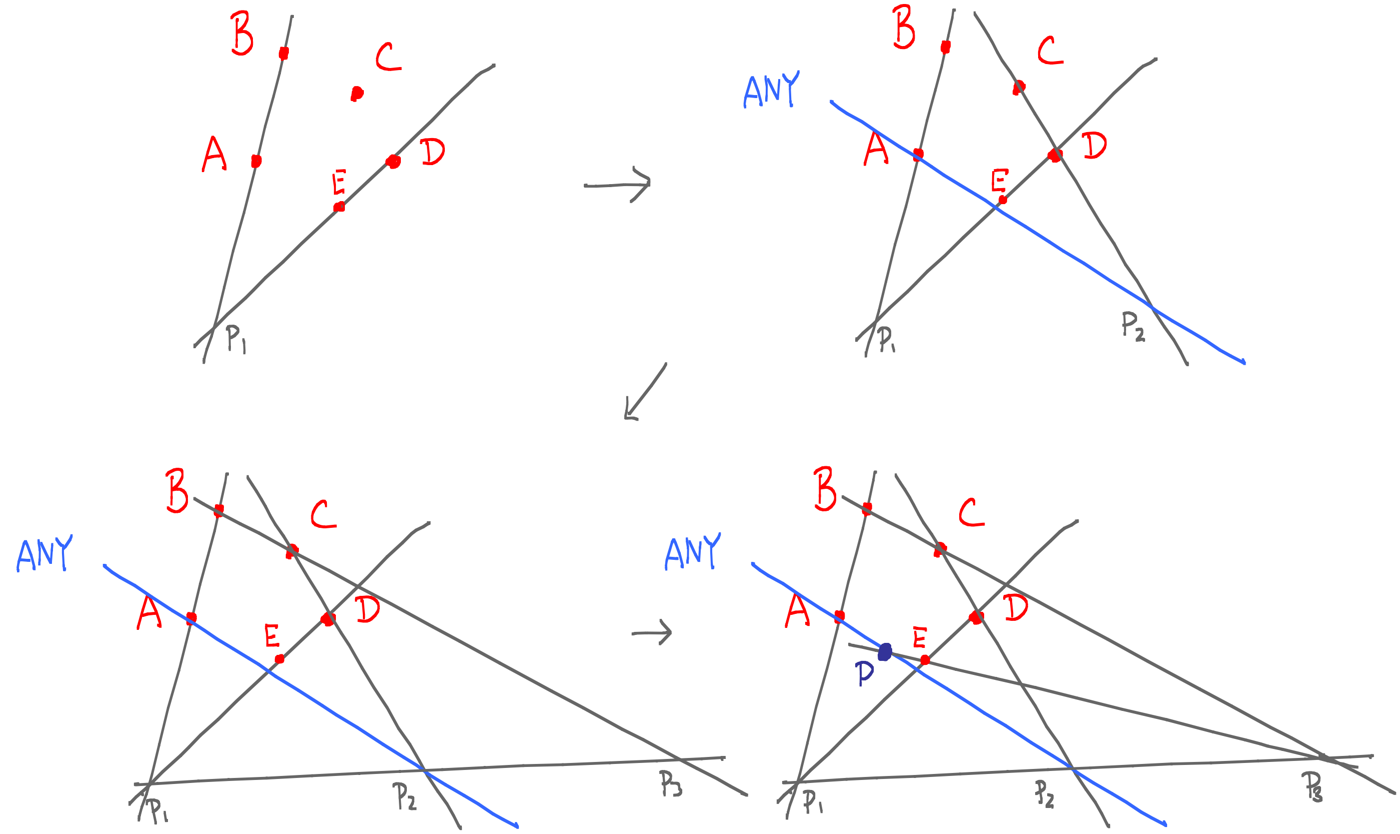
$\Rightarrow f_3 = c_1 f_1 + c_2 f_2$

But $E_1 \cap E_2$ at $3 \times 3 = 9 = 8 + 1$ pt.

Say extra $\pitchfork$-pt. $p_9 \in E_1 \cap E_2$

$\Rightarrow p_9 \in E_3$ $\qquad$ [Cubic Trick].

- Given 5 points $A, B, C, D, E \in \mathbb{C}^2$, how to construct this unique conic Σ?



Claim: $P \in \Sigma$

Pf: i) $\Sigma \cup \overline{P_1P_2}$, $AB \cup CD \cup EP$, $PA \cup BC \cup DE$

3 cubics, all thru. 8 pts: $ABCDEP_1P_2P_3$

ii) $\{\text{cubics}\} \simeq \mathbb{CP}^9$

$\{\text{cubics thru. 8 pts}\} \simeq \mathbb{CP}^1$

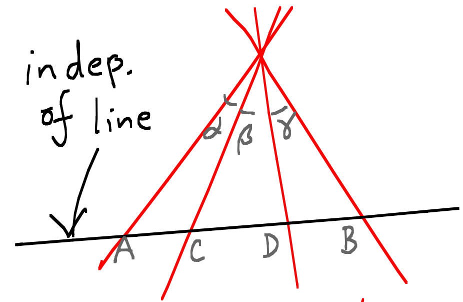
$\Rightarrow$ These 3 cubics are lin. dep.

$P \in$ last 2 $\Rightarrow P \in \Sigma \cup \overline{P_1P_2} \Rightarrow P \in \Sigma$ #

[Cubic Trick].

§ Cross Ratio.

Sine law $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ $\Longrightarrow$



$$\frac{AC/BC}{AD/BD} = \frac{\sin\alpha / \sin(\beta+\gamma)}{\sin(\alpha+\beta)/\sin\gamma}.$$

$\Rightarrow \forall$ distinct $[x_1, y_1], \cdots\cdots, [x_4, y_4] \in \mathbb{CP}^1$

$$\frac{(x_1 y_3 - y_1 x_3)/(x_2 y_3 - y_2 x_3)}{(x_1 y_4 - y_1 x_4)/(x_2 y_4 - y_2 x_4)} =: (P_1, P_2, P_3, P_4)$$

invariant under linear fractional transf.

- Remark: Distance $P_1, P_2 \in S^2 \subset \mathbb{R}^3$

$P_1, P_2 \in L \triangleq \{ P_1 + \lambda P_2 \mid \lambda \in \mathbb{C} \cup \infty \} \subset \mathbb{CP}^2$

$Q \triangleq \{ x^2 + y^2 + z^2 = 0 \} \subset \mathbb{CP}^2$

$L \cap Q = \{ R_1, R_2 \}.$

$\Rightarrow \quad \rho_{S^2}(P_1, P_2) = \frac{\pm 1}{2i} \log (P_1, P_2, R_1, R_2)$

(also work for $K = 1, 0, -1$ surfaces !)

§ Polar curve / Dual curve

$$\Sigma = \{ F(x, y, z) = 0 \} \subseteq \mathbb{CP}^2.$$

$$\rightsquigarrow \quad \mathcal{D}_\Sigma : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2$$

$$[x_0, y_0, z_0] \longmapsto \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]_{(x_0, y_0, z_0)}.$$

- If $P \in \Sigma$, then $\underbrace{\mathcal{D}_\Sigma(P)}_{}= \{\text{tangent line to } \Sigma \text{ at } P\}$

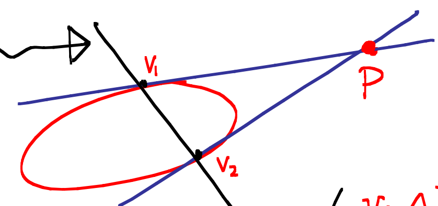
  dual curve $\quad \hat{\Sigma} \subseteq \mathbb{CP}^{2*}$

  Eg: Σ conic : $v A {}^t v = 0$

  $\Rightarrow \hat{\Sigma}$ is $u A^{-1} {}^t u = 0$

  In particular $\hat{\Sigma}$ conic & $\hat{\hat{\Sigma}} = \Sigma$ (always true)

- If $P \notin \Sigma \Rightarrow \mathcal{D}_\Sigma(P) \rightsquigarrow$



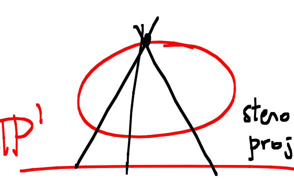
  In particular,

  $$\mathbb{CP}^2 \longleftrightarrow S^2\Sigma$$

  $$P \longleftrightarrow (v_1, v_2) \sim (v_2, v_1)$$

  (i.e. $v_i A {}^t v_i = 0$, $v_i A {}^t P = 0$)

  $$\Rightarrow \mathbb{CP}^2 \cong S^2(\mathbb{CP}^1). \quad (\because \Sigma \simeq \mathbb{CP}^1$$

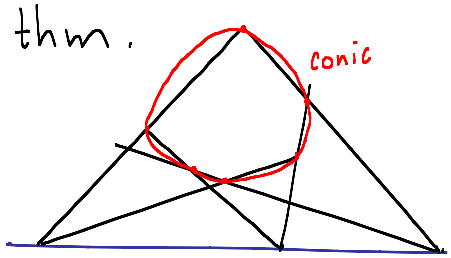


  stenographic projection$)$

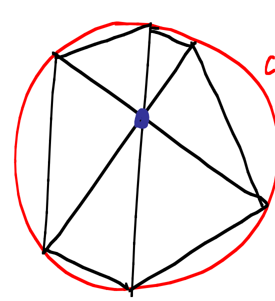
- $\mathcal{D}_\Sigma$: pt. $\longrightarrow$ line

  line $\longrightarrow$ pt.

  eg. Pascal thm.



dual $\rightsquigarrow$ statement

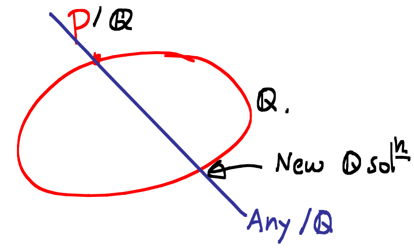


§ Rational Points on Conics.

$v A \, {}^t v = 0$ w/ $A / \mathbb{Q}$

$\overset{?}{\exists}$ sol^n. / $\mathbb{Q}$?

- $\exists\, 1$ $\mathbb{Q}$-sol^n. $P \Rightarrow \exists$ many



- WLOG $\varepsilon_1 x^2 + \varepsilon_2 y^2 - z^2 = 0$

  $|\varepsilon_i|$: product of distinct primes.

- Eg. $3x^2 + y^2 - z^2 = 0$ w/ $\mathbb{Q}$-sol^n. $(1, 1, 2)$

  Eg. $3x^2 + 2y^2 - z^2 = 0$ $\nexists$ $\mathbb{Q}$-sol^n.

  reason: 'IF' (x_0, y_0, z_0) is $\mathbb{Z}$-sol^n. w/o common factor

  (i) $3 \nmid y_0 \Rightarrow$ (※) (via (mod 3) considerations)

  (ii) $3 \mid y_0 \Rightarrow 3 \mid z_0 \Rightarrow 3^2 \mid 3x_0^2 \Rightarrow 3 \mid x_0 \Rightarrow \exists$ common factor (※)

- Similar considerations $\Rightarrow$ determine $\overset{?}{\exists}$ $\mathbb{Q}$-sol^n.

Chapter 2. Cubics.

$$E = \{ F(x,y,z) = 0 \} \subset \mathbb{CP}^2$$
$$\deg F = 3$$

$$\mathcal{D}_E : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2 \qquad \text{NOT } \cong .$$

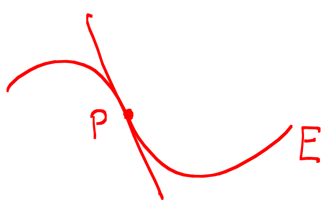
$\mathcal{D}_E$ degen. at P

$$\Longleftrightarrow \quad \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix} (P) = 0 \qquad \leftarrow \text{deg. 3 eqt.}$$

So $\mathcal{D}_E|_E$ degen. at $3 \times 3 = 9$ points, called <u>inflection points</u>.

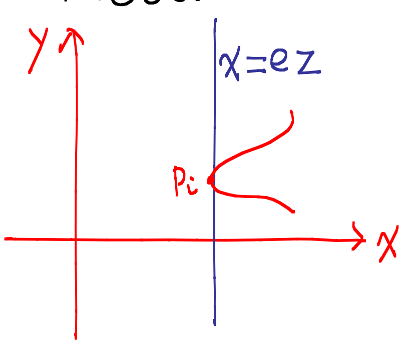
$\Longleftrightarrow$ E & $T_P E$ have contact of order 3 at P.

(tangent $\equiv$ order 2 contact).



Assume $p_\infty = (0,1,0) \in E$ inflection pt.

$T_{p_\infty} E = \{ z = 0 \}$

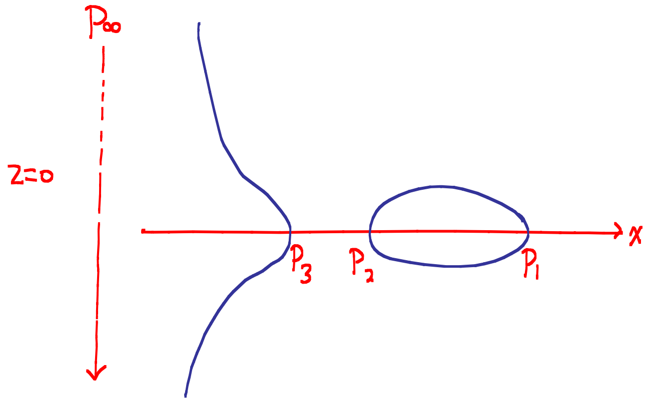


$\exists$ 3 'vertical' tangents (say at p_1, p_2, p_3)
$(x = ez)$

<u>Claim</u>: p_1, p_2, p_3 collinear.

[Pf: "Cubic Trick"]

WLOG: $P_1 = (0\,0\,1)$ $\quad P_2 = (1\,0\,1)$ $\quad P_3 = (a\,0\,1)$



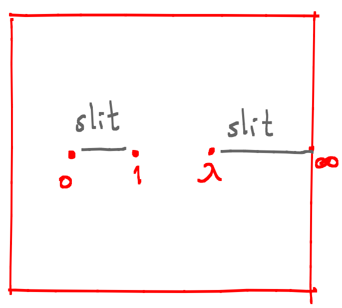
$\rightsquigarrow \quad y^2 - x(x-1)(x-\lambda) = 0$

$\rightsquigarrow$ Stereographic projection

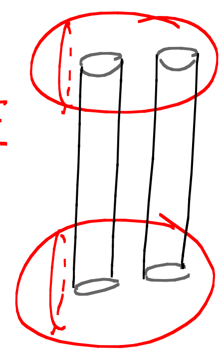
$$\begin{array}{c} E \\ \downarrow \; 2:1 \\ x \in \mathbb{CP}^1 \end{array}$$

Branch at: P_∞, P_1, P_2, P_3

(branch $\Leftrightarrow$ loc. $\begin{array}{ccc}(y^2 = x) & \longrightarrow & (x\text{-axis}) \\ (x,y) & \mapsto & x\end{array}$)



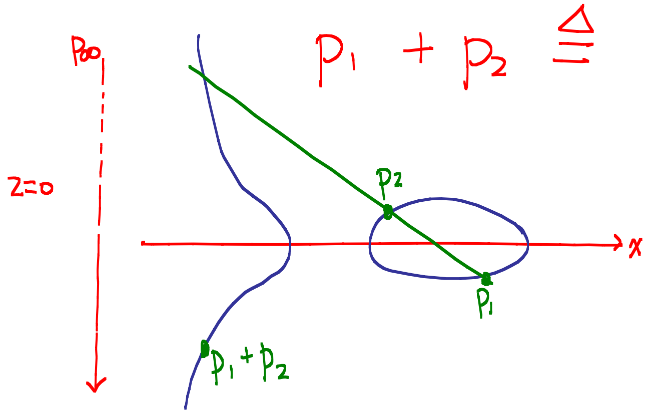
$\begin{array}{c}\mathbb{CP}^1 \\ \rotatebox{90}{$\in$} \\ x\end{array} \xleftarrow{2:1} E$



($\Rightarrow$ E genus 1)

§ Group structure on E.

Choose inflection point $P_\infty \in E$.



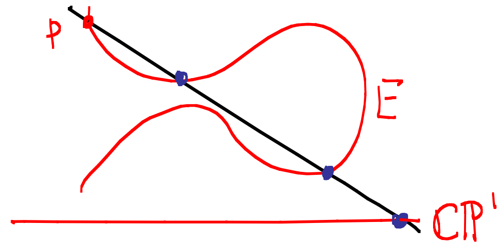
$p_1 + p_2 \overset{\Delta}{=} (x, -y)$ if $p_1, p_2, (x,y)$ collinear

- P_∞ : identity
- commutative (obvious)
- associativity [Cubic Trick].

Remark: Group str. $/\mathbb{Q}$?

- $E/\mathbb{Q} \not\Rightarrow \exists$ inflect pt. $P_\infty/\mathbb{Q}$
- But $\exists$ Cremona transf. (skip) $\rightsquigarrow$ any point (say $/\mathbb{Q}$) becomes inflection point.

- $\forall\ p \in E \subset \mathbb{CP}^2$
  projection from p
  $\rightsquigarrow f : E \xrightarrow[\text{deg } 2]{} \mathbb{CP}^1$



- $\rightsquigarrow$ 4 branch pts. $p_1, p_2, p_3, p_4 \in \mathbb{CP}^1$
  $\rightsquigarrow$ cross ratio $(p_1, p_2, p_3, p_4) \in \mathbb{C} \setminus \{0,1\}$

  Indep. of choice of projection p!
  ($\because$ max. principle).

- Indeed, ALL deg2 $g : E \to \mathbb{CP}^1$ come from projections! ($\because f/g : E \longrightarrow \mathbb{CP}^1$ deg 1 $\rightsquigarrow$ const. map)

  In particular, cross ratio is "intrinsic".

- Cor. $E, E' \overset{\text{cubic}}{\subset} \mathbb{CP}^2$, $E \overset{\text{cx.mfd.}}{\simeq} E'$
  $\Rightarrow$ same cross ratios.

  [$\Leftarrow$] ($\because$ cross ratio $= \lambda \Rightarrow E : y^2 = x(x-1)(x-\lambda)$).

i.e. $\left\{ \begin{array}{l} \text{smooth} \\ \text{cubics} \end{array} \text{in } \mathbb{P}^2 \right\} / \text{isom.} \simeq \mathbb{C} \setminus \{0,1\} \Big/ \lambda \sim \frac{1}{\lambda} \sim 1-\lambda \sim \frac{1}{1-\lambda} \sim \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}$

§ Elliptic integral / period.

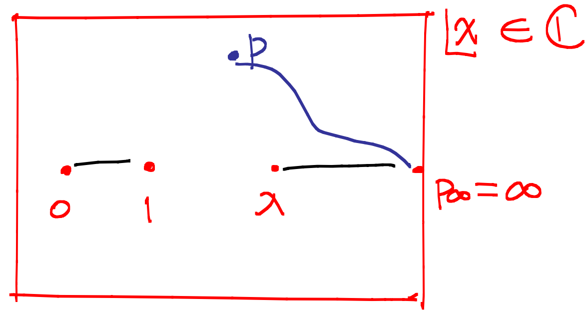
$$C = \{y^2 - x(x-1)(x-\lambda) = 0\}^- \subset \mathbb{CP}^2$$

$$\omega \triangleq \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \in \Omega^{1,0}(C)$$

non-vanishing ($\because$ explicit expansion).

- $E \ni p \longmapsto \int_{(0:1:0) = p_\infty}^{p} \omega \in \mathbb{C}$

- depend on path from p_∞ to p.



- well-defd. modulo

$$\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$$

$$\pi_1 = 2\int_0^1 \omega \qquad \pi_2 = 2\int_1^\lambda \omega$$

- π_1, π_2 : linearly indep. $/\mathbb{R}$

because $(\mathrm{Re}\,\pi_1)(\mathrm{Im}\,\pi_2) - (\mathrm{Im}\,\pi_1)(\mathrm{Re}\,\pi_2) > 0$ Riemann relat$^{\underline{n}}$.

($\because$ $\omega \in \Omega^{1,0} \Rightarrow i\int_C \omega \wedge \bar{\omega} > 0$)

(eg. $\pi_1 = 1$, $\pi_2 =: \tau \Rightarrow \mathrm{Im}\,\tau > 0$)

- $\int_\infty^{\cdot} : E \xrightarrow{\cong} \mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$

- Dependence on λ (Picard-Fuchs eqt.)

$$\pi_1(\lambda) := 2\int_0^1 \omega$$

$$\pi_2(\lambda) := 2\int_1^\lambda \omega$$

Explicitly calculation:

$$\frac{1}{4}\omega + (2\lambda-1)\frac{\partial\omega}{\partial\lambda} + \lambda(\lambda-1)\frac{\partial^2\omega}{\partial\lambda^2} = d(\text{sth.})$$

$$((\text{sth})= \tfrac{-1}{2} y\,(x-\lambda)^{-2})$$

$$\Rightarrow \frac{1}{4}\pi_i + (2\lambda-1)\frac{d\pi_i}{d\lambda} + \lambda(\lambda-1)\frac{d^2\pi_i}{d\lambda^2} = 0, \quad i=1,2$$

ODE w/ regular singular points.

$\Rightarrow$ Space of sol^{n}. near $\lambda=0$ gen. by

$\sigma_1(\lambda)$ & $\lambda\sigma_2(\lambda) + (\log\lambda)\,\sigma_1(\lambda)$

where σ_i's holo. & non-vanish at $\lambda=0$

Check: $\sigma_1(\lambda) = \dfrac{\pi_1(\lambda)+\pi_2(\lambda)}{2\pi} = \dfrac{2\int_0^\lambda \omega}{2\pi}$

$\sigma_1(0) = 1.$ (residue calculation).

- Power series method $\Rightarrow$

$$\sigma_1(\lambda) = \sum_{n=0}^\infty a_n\lambda^n \qquad a_{n+1} = \left|\frac{n+\frac{1}{2}}{n+1}\right|^2 a_n$$

$$\Rightarrow \quad = \sum_{n=0}^\infty \binom{-1/2}{n}^2 \lambda^n.$$

Note: $\pi_1(\lambda) = \int_0^1 \omega \sim \log\lambda$ near $\lambda=0$.

§ Rational points on $C/\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$

$$C_\lambda = \{ y^2 = x(x-1)(x-\lambda) \} \subseteq \mathbb{F}_p^2 \quad \text{assume } \lambda \in \mathbb{Z}$$

<u>Claim</u>: $\# C_\lambda \equiv (-1)^{(p-1)/2+1} \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r}^2 \lambda^r \pmod p$

SAME formula!!

Pf. $\# C_\lambda \sim \# \{ x \in \mathbb{F}_p : x(x-1)(x-\lambda) \text{ is square} \}$

$$\text{i.e.} \begin{cases} [x(x-1)(x-\lambda)]^{\frac{p-1}{2}} \equiv 1 & \Rightarrow 2 \text{ sol}^{\underline{n}} \\ \text{or} \quad x = 0, 1, \lambda & \Rightarrow 1 \text{ sol}^{\underline{n}} \end{cases}$$

(otherwise $[\text{ —}''\text{— }]^{\frac{p-1}{2}} \equiv -1$)

$$\Rightarrow \# C_\lambda = \sum_{x \in \mathbb{F}_p} \left(1 + [\text{ —}''\text{— }]^{\frac{p-1}{2}} \right)$$

$$= \text{coeff. of } x^{p-1} \text{ in } \sum_{x \in \mathbb{F}_2} [\text{ —}''\text{— }]^{\frac{p-1}{2}}$$

$$\left(\because \sum_{x \in \mathbb{F}_p} x^k = \begin{cases} 0 & (p-1) \nmid k \\ -1 & (p-1) \mid k \end{cases} \right)$$

$$\overset{\text{expansion}}{=} (-1)^{(p-1)/2} \sum_{r=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{r}^2 \lambda^r$$

$$\equiv (-1)^{(p-1)/2} \sum_{r=0}^{(p-1)/2} \binom{-\frac{1}{2}}{r}^2 \lambda^r \pmod p.$$

$$\equiv (-1)^{(p-1)} \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r}^2 \lambda^r \pmod p.$$

$(\because r \geq (p+1)/2 \Rightarrow \binom{-\frac{1}{2}}{r} \equiv 0 \pmod p).$ #

§ "Explanation"

Recall: Lefschetz fix pt. formula

$f: M \to M$ cpt. oriented mfd.

$\rightsquigarrow f^*: H^k(M) \to H^k(M) \quad \forall k$

Thm. $$\underbrace{\# \operatorname{Fix}(f)}_{\substack{\| \\ \sum_{f(x)=x} (\pm 1)}} = \sum_k (-1)^k \operatorname{Tr}_{H^k} f^*$$

$(\pm 1) \leftarrow \operatorname{sgn} \underbrace{\det(I - \overbrace{J(f)_x}^{\text{Jacobian}})}_{\sum_{r=0}^{n} (-1)^r \operatorname{Tr} \Lambda^r J(f)_x}$.

$$\underbrace{}_{\sum_{\substack{f(x)=x \\ r}} (-1)^r \frac{\operatorname{Tr} \Lambda^r J(f)_x}{|\det(I - J(f)_x)|}}$$

Holom. Lef. formula: M Kähler + $\bar\partial f = 0$

(replace deRham cpx. $d = \partial + \bar\partial : \Omega^{\bullet} ↺$ by Dolbeault cpx. $\bar\partial : \Omega^{0,\bullet} ↺$)

$$\underbrace{\sum_{\substack{f(x)=x \\ r}} (-1)^r \frac{\operatorname{Tr} \Lambda^r J^{01}(f)_x}{|\det(I - J(f)_x|}}_{\sum_{f(x)=x} \frac{1}{\det(I - J^{10}(f)_x)}} = \sum_k (-1)^k \operatorname{Tr}_{H^k(M,\mathcal{O})} f^*$$

FACT : work / $\overline{\mathbb{F}_p}$

$$M := \{ y^2 z = x(x-z)(x-\lambda z) \} \subseteq \overline{\mathbb{F}_p}\, \mathbb{P}^2$$

$f \circlearrowleft$ $f(x,y,z) = (x^p, y^p, z^p)$ Frobenius map

$$\sum_{\substack{f(x)=x \\ \updownarrow \\ x \in \mathbb{F}_p}} \frac{1}{\det(1 - \cancel{J^{10}(f)_x})} = \underbrace{\sum_k (-1)^k \mathrm{Tr}_{H^k(M,\mathcal{O})} f^*}_{1 - \mathrm{Tr}_{H^1(M,\mathcal{O})} f^*}$$

$\left(\because \frac{dx^p}{dx} = px^{p-1} \equiv 0 \pmod p\right)$ $\quad \left(\because H^{>1} = 0 \atop \dim M = 1\right)$

$$\Rightarrow \quad \# C_\lambda = - \mathrm{Tr}_{H^1(M,\mathcal{O})} f^*$$

Serre duality,

both 1 dim.

$$H^1(M,\mathcal{O}) \otimes H^0(\Omega^1(M)) \xrightarrow{\text{perfect pairing}} \overline{\mathbb{F}_p}$$

$$h \otimes \omega \longmapsto \mathrm{Res}_q(h\omega).$$

$n \to \infty$, $\quad 0 \to \mathcal{O} \to \mathcal{O}(nq) \oplus \mathcal{O}(nq') \to \mathcal{O}(nq + nq') \to 0$

$$\Rightarrow \quad H^1(M,\mathcal{O}) = \frac{H^0(\mathcal{O}(nq+nq'))}{H^0(\mathcal{O}(nq)) + \underbrace{H^0(\mathcal{O}(nq'))}_{}}$$

$\leftarrow$ alg. fu. w/ poles at q' (of order at most $n \to \infty$)

RR $\Rightarrow$ $\exists\, h$ w/ poles at q, q'

simple pole at q.

write $$h(x) = \frac{1}{x - x(q)} + \sum_{\ell \geq 0} b_\ell (x - x(q))^\ell$$

$$h(x^p) = \frac{1}{(x - x(q))^p} + \sum b_\ell (x - x(q))^{\ell p}$$

$$\Rightarrow \quad \mathrm{Tr}_{H^1(\mathcal{O})} f^* = \text{coeff. of } \frac{1}{x - x(q)} \text{ in } h(x^p)\omega.$$

$$= a_{p-1}(\lambda)$$

$$\omega = dx + \sum_{r \geq 1} a_r(\lambda)(x - x(q))^r dx$$

PF eqt:

$$\left(\lambda(\lambda-1)\frac{\partial^2}{\partial\lambda^2} + (2\lambda-1)\frac{\partial}{\partial\lambda} + \frac{1}{4}\right)\left(1 + \sum a_r(\lambda)(x - x(q))^r\right) = \frac{d}{dx}\left(\frac{x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-\lambda)^{\frac{1}{2}}}{(x-\lambda)^2}\right)$$

$$\Rightarrow \left(\text{———— ,, ————}\right) a_{p-1}(\lambda)(x - x(q))^{p-1} = \frac{d}{dx}\left(c(\lambda)(x - x(q))^p\right) \equiv 0$$

$\Rightarrow \quad a_{p-1}(\lambda)$ s.t. PF eqt. (& finite at $\lambda = 0$,)

$$\Rightarrow \quad a_{p-1}(\lambda) = c \sum \binom{-\frac{1}{2}}{r}^2 \lambda^r \qquad \exists\, c \text{ const.}$$

(✓).

#.

Chapter 3. Theta functions.

§ smooth cubic $E \subset \mathbb{CP}^2$

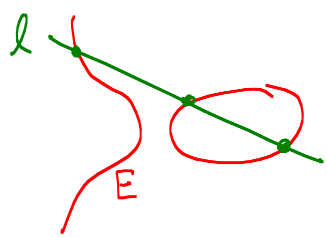
$\rightsquigarrow \quad \pi_1 = \int_{\gamma_1} \omega \ \ \& \ \ \pi_2 = \int_{\gamma_2} \omega$ period
$H_1(E,\mathbb{Z}) = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$

$$\rightsquigarrow \quad f : E \longrightarrow \mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$$
$$p \longmapsto \int_{p_0}^{p} \omega$$

<u>Claim:</u> f group homo.

Pf: Recall $p_1 + p_2 + p_3 = 0$ in E

$\Leftrightarrow \quad \{p_1, p_2, p_3\} = E \cap \ell \quad \exists$ line $\ell \subset \mathbb{CP}^2$
(i.e. $\ell \in \mathbb{CP}^{2*}$)



$$\mathbb{CP}^{2*} \longrightarrow \mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$$
$$\ell \longmapsto \sum_{p \in \ell \cap E} f(p)$$

$$\pi_1(\mathbb{CP}^2) = 0 \Rightarrow \begin{tikzcd} & \mathbb{C} \arrow[d] \\ \mathbb{CP}^{2*} \arrow[ur, dashed, "\exists \text{ lift}"] \arrow[r] & \mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2 \end{tikzcd} \overset{\text{max. pr.}}{\Longrightarrow} \text{const. map}$$

Choose $\ell = \{z = 0\}$
$\ell \cap E = 3p_\infty$ (order 3 contact)

Also $f(p_\infty) = 0$ ($\because$ integrat^n path shrinks)

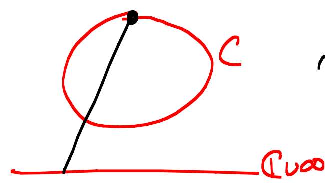
Hence $p_1 + p_2 + p_3 = 0 \Leftrightarrow f(p_1) + f(p_2) + f(p_3) = 0$ #

Every smooth cubic $E \subset \mathbb{CP}^2$ is $\mathbb{C}/\Lambda$.

Qu: Converse ?

Ans: Yes. (prove later).

§ $C \underset{\text{conic}}{\subset} \mathbb{CP}^2$



$\rightsquigarrow$ $C \xrightarrow{\sim} \mathbb{CP}^1$

Indeed $\mathbb{C} \longrightarrow C$ is algebraic.

<u>Claim</u> $\mathbb{C} \longrightarrow E \underset{\text{cubic}}{\subset} \mathbb{CP}^2$ Never alg!

$$E = \{y^2 = x(x-1)(x-\lambda)\}$$

i.e. $\nexists$ $f(z) = \dfrac{p_1(z)}{p_2(z)}$ & $g(z) = \dfrac{q_1(z)}{q_2(z)}$ (nonconst)

p_1, p_2 (resp. q_1, q_2) rel. prime polyn.

s.t. $g^2 = f(f-1)(f-\lambda)$.

Pf: Otherwise,

$$\left(\frac{q_1}{q_2}\right)^2 = \frac{p_1}{p_2}\left(\frac{p_1}{p_2} - 1\right)\left(\frac{p_1}{p_2} - \lambda\right)$$

i.e. $p_2^3 q_1^2 = q_2^2 \, p_1 (p_1 - p_2)(p_1 - \lambda p_2)$

$\Rightarrow$ $p_2^3 \mid q_2^2$ & $q_2^2 \mid p_2^3$

$\Rightarrow$ $\begin{cases} p_2^3 = c\, q_2^2 \\ q_1^2 = p_1(p_1 - p_2)(p_1 - \lambda p_2) \end{cases}$

$\Rightarrow$ $p_1, \; p_2, \; p_1 - p_2, \; p_1 - \lambda p_2$ perfect square

write $p_1 = r_1^2$, $p_2 = r_2^2$

$$\Rightarrow \quad \underbrace{r_1^2 - r_2^2}_{(r_1-r_2)(r_1+r_2)} = \text{sq.} \qquad \underbrace{r_1^2 - \lambda r_2^2}_{(r_1-\sqrt{\lambda}r_2)(r_1+\sqrt{\lambda}r_2)} = \text{sq.}$$

$$\Rightarrow \quad r_1 - r_2, \; r_1 + r_2, \; r_1 - \sqrt{\lambda} r_2, \; r_1 + \sqrt{\lambda} r_2 \quad \text{perfect square}$$

$\cdots\cdots \Rightarrow$ can take square root ∞ times $(\longrightarrow\!\!\!\!\!\!\times\;)$

Remark: Unless $\lambda = 0$ or 1

i.e. singular cubic.

Indeed $\mathbb{C} \longrightarrow \{ y^2 = x^2(x-1) \}$ alg!

$a \mapsto (a^2+1, \; a(a^2+1))$

§ $E \simeq \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau$, $\operatorname{Im}\tau > 0$.

$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad \bar{\partial} f = 0$

$$\left. \begin{array}{l} f(u+1) = f(u) \\ f(u+\tau) = f(u) \end{array} \right\} \Rightarrow \text{decend } f : E \longrightarrow \mathbb{C} \Rightarrow \text{const.}$$

Keep $f(u+1) = f(u) \overset{\text{Fourier}}{\Rightarrow} f(u) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n u}$

Replace $f(u+\tau) = e^{-2\pi i (u+\alpha)} f(u)$

$\Rightarrow \quad a_{n+1} = a_n e^{2\pi i (n\tau + \alpha)} \qquad$ (say $a_0 = 1$)

Take $\alpha = \frac{\tau}{2} \Rightarrow a_n = e^{2\pi i \sum_{k=1}^{n} (k - \frac{1}{2}) \tau} = e^{\pi i n^2 \tau}$

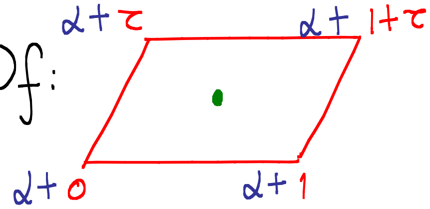
$$\Rightarrow f(u) = \underline{\sum_{n=-\infty}^{+\infty} e^{\pi i (n^2 \tau + 2 n u)} =: \vartheta\begin{bmatrix}0\\0\end{bmatrix}(u;\tau)}$$

(cgt. ✓) $\qquad$ ($\sim$ section of bdl./E)

- Even function in u.

Claim. ϑ has a unique (simple) zero at $\frac{1}{2}+\frac{\tau}{2}$

Pf:



$$\frac{1}{2\pi i}\int_{\alpha+\square} d\log\vartheta = \frac{1}{2\pi i}\int_{\alpha+\tau+1}^{\alpha+\tau}(-2\pi i)\,du = 1$$

$\Rightarrow \exists!$ zero. Where ?

$$\frac{1}{2\pi i}\int_{\alpha+\square} u\, d\log\vartheta = \cdots\cdots = \frac{1}{2}+\frac{\tau}{2} + m + n\tau \quad \text{w/ } m,n\in\mathbb{Z}$$

$$\vartheta(u+1)=\vartheta(u), \quad \vartheta(u+\tau)=e^{-2\pi i(u+\frac{\tau}{2})}\vartheta(u)$$

$$\Rightarrow \quad f(u) \triangleq \frac{\prod_j \vartheta(u-p_j-(\frac{1}{2}+\frac{\tau}{2}))}{\prod_j \vartheta(u-q_j-(\frac{1}{2}+\frac{\tau}{2}))} \qquad \underbrace{\text{same \# of } p\text{'s} \ \& \ q\text{'s}}_{\Downarrow}$$

satisfies $f(u+1)=f(u)$ & $f(u+\tau)=e^{2\pi i(\sum p_j-\sum q_j)}f(u)$

w/ zero at p's, poles at q's.

IF $\sum p_j = \sum q_j$ on E

(i.e. $\sum p_j = \sum q_j + m + n\tau$ on $\mathbb{C}$)

$$\Rightarrow \quad e^{-2\pi i n u}\frac{\prod_j \vartheta(u-p_j-(\frac{1}{2}+\frac{\tau}{2}))}{\prod_j \vartheta(u-q_j-(\frac{1}{2}+\frac{\tau}{2}))}$$

mero. fu. of E w/ zeros p's & poles q's.

Abel Theorem $(g=1)$

$$\exists f: E\longrightarrow \mathbb{CP}^1 \iff \sum_{f(p)=0} p = \sum_{f(q)=\infty} q + m + n\tau \quad \exists\, m,n\in\mathbb{Z}$$

Similar argument $\Rightarrow$

Riemann-Roch $(g=1)$

$r_1+\cdots+r_s \geq 2$ $\quad r_j>0$, any $q_j\in E$

$$\Rightarrow \dim\{f: E\to\mathbb{CP}^1 \mid f^{-1}(\infty)\subseteq \sum_j r_j q_j\} = \sum_j r_j$$

$u \in \mathbb{C}$, $\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u;\tau) \triangleq \sum_{n=-\infty}^{\infty} e^{\pi i[(n+\frac{1}{2})^2\tau + 2(n+\frac{1}{2})(u+\frac{1}{2})]}$

- cgt. ✓
- odd (switch $n \to -n$) (w/ zero at $\mathbb{Z}+\tau\mathbb{Z}$)
- $\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u+1,\tau) = -\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u,\tau)$

  $\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u+\tau,\tau) = -e^{-\pi i(\tau+2u)}\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u,\tau)$

$\Rightarrow$ $\frac{d^2}{du^2}\log\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u,\tau)$ doubly periodic (i.e. descend to E)

double pole at 0 $\left(\because \frac{d^2}{du^2}\log u = \frac{-1}{u^2}\right)$

$\xRightarrow[\text{at } 0]{\text{Laurent}}$ — " — $= \frac{-1}{u^2} + F(u^2)$ around 0.

Similarly $\frac{d^2}{du^2}\log\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\tau u,\tau)$ & $\frac{d^2}{du^2}\log\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u,\frac{-1}{\tau})$ doubly periodic wrt $\mathbb{Z}+\frac{-1}{\tau}\mathbb{Z}$

around 0: $\frac{-1}{u^2}+C+F(u^2)$

$\xRightarrow{RR}$ (— " —) = (— " —) + $C(\tau)$

$\Longrightarrow$ $\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u,\frac{-1}{\tau}) = \alpha\, e^{\beta u^2+\gamma u}\,\vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](u\tau,\tau)$ (w/ α,β,γ: fu. of τ)

- $\gamma(\tau) = 0$ ($\because$ ϑ: odd fu. in u)
- $\left.\begin{matrix}u\to u+1\\ u\to u+\tau\end{matrix}\right\} \Rightarrow 1 = e^{\beta(2u+1)}e^{-\pi i\tau(2u+1)} \Rightarrow \beta(\tau) = \pi i\tau$
- (set $u=0$) $\sum e^{\pi i n^2(\frac{-1}{\tau})} = \alpha_0(\tau)\sum e^{\pi i n^2\tau}$

$f(t) = e^{-\pi x t^2} \xRightarrow[\text{Fourier}]{\text{(complete }\square)} \hat{f}(s) = x^{-\frac{1}{2}}e^{-\pi s^2/x}$

Poisson summation formula $\Rightarrow$ $\sum_n e^{-\pi x n^2} = x^{-\frac{1}{2}}\sum_n e^{-\pi n^2/x}$

$\Rightarrow$ $\alpha_0(ix) = x^{\frac{1}{2}}$ (for $\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$)

$\Rightarrow$ $\vartheta\left[\begin{smallmatrix}\delta\\\varepsilon\end{smallmatrix}\right](u,\frac{-1}{\tau}) = \left(\frac{\tau}{i}\right)^{\frac{1}{2}}(-i)^{\delta\varepsilon}e^{\pi i u^2\tau}\,\vartheta\left[\begin{smallmatrix}\varepsilon\\\delta\end{smallmatrix}\right](u\tau,\tau)$

$\tau \mapsto \tau+1$
Easy

$\vartheta\left[\begin{smallmatrix}1\\\varepsilon\end{smallmatrix}\right](u,\tau+1) = e^{\pi i/4}\,\vartheta\left[\begin{smallmatrix}1\\\varepsilon\end{smallmatrix}\right](u,\tau)$

$\vartheta\left[\begin{smallmatrix}0\\\varepsilon\end{smallmatrix}\right](u,\tau+1) = \vartheta\left[\begin{smallmatrix}0\\1-\varepsilon\end{smallmatrix}\right](u,\tau)$,

$$\frac{d^2}{du^2}\log\vartheta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau) \overset{\text{around } 0}{=} \frac{1}{u^2} + C_0(\tau) + C_2(\tau)u^2 + \cdots\cdots$$

- $\lim_{\tau\to i\infty}\left(\text{———} \;\;\text{''}\;\; \text{———} \right) = \pi^2\csc^2(\pi u)$

  $\left(\because \lim_{\tau\to i\infty} e^{-\pi i\tau/4}\vartheta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau) = e^{\pi i(u+\frac12)} + e^{-\pi i(u+\frac12)} = -2\sin\pi u\right)$

- $C_{2n}(\tau+1) = C_{2n}(\tau)$ & $C_{2n}\left(\frac{-1}{\tau}\right) = \tau^{2n+2}C_{2n}(\tau)$

  bound as $\tau\to i\infty$. (i.e. modular forms of wt. $n+1$!).

Remark: $\nexists$ wt. 1 modular forms $\} = 0$

wt 2 $\underset{\text{up to const.}}{\Longleftrightarrow}$ $C_2(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0}\frac{1}{(m+n\tau)^4}$

wt 3 $\Longleftrightarrow$ $C_4(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0}\frac{1}{(m+n\tau)^6}$

Claim: At $u=0$,

(1) $\vartheta\begin{bmatrix}0\\0\end{bmatrix}^8 + \vartheta\begin{bmatrix}1\\0\end{bmatrix}^8 + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^8 = \frac{30}{\pi^4}C_2(\tau)$

(2) $\left(\vartheta\begin{bmatrix}0\\1\end{bmatrix}^4 + \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4\right)\left(\vartheta\begin{bmatrix}1\\0\end{bmatrix}^4 + \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4\right)\left(\vartheta\begin{bmatrix}0\\1\end{bmatrix}^4 - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4\right) = \frac{189}{\pi^6}C_4(\tau)$

Pf: For (1), LHS is modular form of wt 2.

$$\vartheta\begin{bmatrix}\delta\\\varepsilon\end{bmatrix}\left(u,\frac{-1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{\frac12}(-i)^{\delta\varepsilon}e^{\pi i u^2\tau}\,\vartheta\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}(u\tau,\tau)$$

$$\vartheta\begin{bmatrix}1\\\varepsilon\end{bmatrix}(u,\tau+1) = e^{\pi i/4}\,\vartheta\begin{bmatrix}1\\\varepsilon\end{bmatrix}(u,\tau)$$

$$\vartheta\begin{bmatrix}0\\\varepsilon\end{bmatrix}(u,\tau+1) = \vartheta\begin{bmatrix}0\\1-\varepsilon\end{bmatrix}(u,\tau),$$

At $u=0$, they becomes

$$\left.\begin{aligned}
&\vartheta\begin{bmatrix}\delta\\\varepsilon\end{bmatrix}\left(\frac{-1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{\frac12}(-i)^{\delta\varepsilon}\,\vartheta\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}(\tau)\\
&\vartheta\begin{bmatrix}1\\\varepsilon\end{bmatrix}(\tau+1) = e^{\pi i/4}\,\vartheta\begin{bmatrix}1\\\varepsilon\end{bmatrix}(\tau)\\
&\vartheta\begin{bmatrix}0\\\varepsilon\end{bmatrix}(\tau+1) = \vartheta\begin{bmatrix}0\\1-\varepsilon\end{bmatrix}(\tau),
\end{aligned}\right\}\Rightarrow
\begin{aligned}
&\vartheta\begin{bmatrix}\delta\\\varepsilon\end{bmatrix}^8\left(\frac{-1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{\frac12 \cdot 8 \to \tau^4}\cancel{(-i)^{8\delta\varepsilon}}\,\vartheta\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}^8(\tau)\\
&\vartheta\begin{bmatrix}1\\\varepsilon\end{bmatrix}^8(\tau+1) = \cancel{e^{8\pi i/4}}\,\vartheta\begin{bmatrix}1\\\varepsilon\end{bmatrix}^8(\tau)\\
&\vartheta\begin{bmatrix}0\\\varepsilon\end{bmatrix}^8(\tau+1) = \vartheta\begin{bmatrix}0\\1-\varepsilon\end{bmatrix}^8(\tau),
\end{aligned}$$

$\mathrm{LHS}(\tau = i\infty) = 2 \Rightarrow (1)$ $(\because \exists$ such mod. form$)$

(2) is similar #

Weierstrass $\wp$-function.

$$\wp(u) \triangleq -\frac{d^2}{du^2}\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau) - c_0(\tau) = \frac{1}{u^2} + 0 + F(u^2) : E \to \mathbb{P}^1$$

$$1,\ \wp,\ \wp',\ \wp^2,\ \wp\wp',\ \wp^3,\ \wp'^2 : E \longrightarrow \mathbb{CP}^1$$

(only) pole at 0, order $=$ $\quad 0 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 6$

RR $\Longrightarrow$ linear dep. (coeff ✓ via Laurent exp. at 0)

$$\wp'^2 = 4\wp^3 - 20c_2\wp - 28c_4$$

$$\Longrightarrow \quad \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}} \longrightarrow E \subseteq \mathbb{CP}^2$$

$$u \longmapsto (\wp, \wp', 1)$$

w/ $E : y^2 = 4x^3 - 20c_2x - 28c_4$

$$0 \longmapsto (0, 1, 0)$$

- In particular, every $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ is a cubic curve.
- This is inverse to $q \longmapsto \int_{x(q_0)}^{x(q)} \frac{dx}{\sqrt{4x^3-20c_2x-28c_4}} : E \to \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$

$\vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right](u,\tau)$ $\quad \delta, \varepsilon = 0$ or 1

$$\begin{cases} \vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right](u+1,\tau) = (-1)^{\delta}\cdot \vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right](u,\tau) \\ \vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right](u+\tau,\tau) = (-1)^{\varepsilon} e^{-\pi i(\tau+2u)}\, \vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right](u,\tau) \end{cases}$$

Explicitly, $\vartheta\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right](u,\tau) = \vartheta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](u+\tfrac{1}{2},\tau) = \sum e^{\pi i[n^2\tau + 2n(u+\frac{1}{2})]}$

$\vartheta\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right](u,\tau) = \vartheta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](u-\tfrac{1}{2},\tau) = \sum e^{\pi i[(n+\frac{1}{2})^2\tau + 2(n+\frac{1}{2})u]}$

Consider, $\quad h: E = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \longrightarrow \mathbb{CP}^1$

$h(u) = \left[\vartheta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](u,\tau)^2, \ \vartheta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](u,\tau)^2\right]$

- well-defined. $\quad$ ($\because$ same transf. under $u \mapsto u+\tau$)
- $h(\frac{1}{2}+\frac{\tau}{2}) = [0, 1]$ $\quad$ & simple branch
- $h(0) = [1, 0]$ $\quad$ —— " ——

  & no other preimage $\Rightarrow$ double cover.
- Other branches pt. : $u = \frac{1}{2}$ & $\frac{\tau}{2}$

(Reason: branch pt. $\longleftrightarrow$ 2 torsion pt.
$[\because \mathbb{CP}^1 \to E, \ q \mapsto \sum_{h(p)=q} p$ is const. map.$]$)

i.e. $\underbrace{\left[\vartheta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\tfrac{1}{2},\tau)^2, \vartheta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](\tfrac{1}{2},\tau)^2\right]}_{\left[\vartheta\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]^2, \ \vartheta\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]^2\right]}$ & $\underbrace{\left[\vartheta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\tfrac{\tau}{2},\tau)^2, \vartheta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](\tfrac{\tau}{2},\tau)^2\right]}_{\left[-\vartheta\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]^2, \ \vartheta\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]^2\right]}$

(at $u=0$)

$\Rightarrow \quad E = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \cong y^2 = x(x-1)(x-\lambda)$

w/ $\lambda = -\vartheta\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]^4 / \vartheta\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]^4$

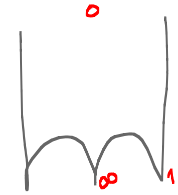
Remark: In fact, all $\lambda, \bar{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}$ are in terms of $\vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right]$'s.

A choice $\longleftrightarrow$ ordered base of $H_1(E, \mathbb{Z}_2)$

level 2 structure.

$\sim$ useful for describing moduli space of E's.

$$-\frac{\vartheta\left[{1 \atop 0}\right]^4}{\vartheta\left[{0 \atop 1}\right]^4} = \begin{cases} 0 & \tau = i\infty \\ \infty & \quad i0^+ \\ 1 & \quad 1+i0^+ \end{cases}$$



($\because$ transf. $\tau \mapsto \tau+1, \frac{-1}{\tau}$)

Similarly, $$\frac{\vartheta\left[{0 \atop 0}\right]^4}{\vartheta\left[{0 \atop 1}\right]^4} = \begin{cases} 1 \\ \infty \\ 0 \end{cases} \Rightarrow 1-\left(-\frac{\vartheta\left[{1 \atop 0}\right]^4}{\vartheta\left[{0 \atop 1}\right]^4}\right) = \frac{\vartheta\left[{0 \atop 0}\right]^4}{\vartheta\left[{0 \atop 1}\right]^4}$$

i.e. $\underline{\vartheta\left[{0 \atop 1}\right]^4 + \vartheta\left[{1 \atop 0}\right]^4 = \vartheta\left[{0 \atop 0}\right]^4}$ Riemann's theta relation.

Similar arguments $\Rightarrow$ Jacobi identity

$$\frac{\partial}{\partial u}\Big|_{u=0} \theta\left[{1 \atop 1}\right](u,\tau) = -\pi\, \vartheta\left[{0 \atop 0}\right] \vartheta\left[{0 \atop 1}\right] \vartheta\left[{1 \atop 0}\right].$$

Chapter 4 Jacobian Variety

§ Complex line bundles.

$$\mathbb{C} \longrightarrow L \longrightarrow M$$

$\sim$ gluing fn. $g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^\times$

s.t. $g_{ij} \cdot g_{jk} \cdot g_{ki} = 1$ on $U_i \cap U_j \cap U_k$

$\sim$ 1-cocycle $[g] \in H^1(M, \mathbb{C}^\times_{cts.})$

i.e. $H^1(M, \mathbb{C}^\times_{cts}) = \{\text{topo. cpx. line bdl}/M\}/\cong$

Consider $0 \to \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1$

$\leadsto$ long exact seq. in cohom.

$$H^1(\mathbb{Z}) \longrightarrow \cancel{H^1(\mathbb{C}_{cts})} \longrightarrow H^1(\mathbb{C}^\times_{cts}) \xrightarrow[c_1]{\cong} H^2(\mathbb{Z}) \longrightarrow \cancel{H^2(\mathbb{C}_{cts})}$$

($\because$ partition of 1)

In particular, c_1 classifies cx. line bdl. !

Similarly, for <u>holom.</u> line bundle L, $\bar\partial g_{ij} = 0$

$\leadsto$ 1-cocycle in $H^1(M, \mathbb{C}^\times_{hol}) =: \mathrm{Pic}(M) = \{\text{holo. line bdl}\}/\cong$

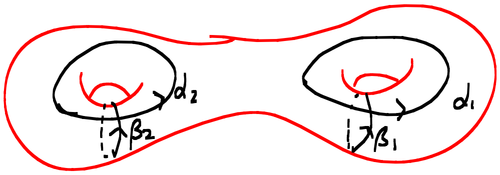
$$\begin{array}{c} \\ H^1(\mathbb{Z}) \longrightarrow H^1(\mathcal{O}) \longrightarrow \end{array} \begin{array}{c} \| \\ H^1(\mathcal{O}^\times) \end{array} \begin{array}{c} \\ \xrightarrow{c_1} H^2(\mathbb{Z}) \end{array}$$

lattice if M cpt. kähler

$\Rightarrow$ $\mathrm{Pic}^0(M) = H^1(\mathcal{O}) / H^1(\mathbb{Z})$ connected component of Pic.

$= \{\text{top. trivial holo. line bdl.}/M\}/\cong$

§ Complex curves C (i.e. compact complex mfd. of dim 1).



$H_1(C,\mathbb{Z}) = \mathbb{Z}\langle \alpha_j\text{'s}, \beta_j\text{'s}\rangle$

$\alpha_j \cdot \alpha_k = 0 = \beta_j \cdot \beta_k$

$\alpha_j \cdot \beta_k = \delta_{jk}$

Hodge theory

$$H^1(C,\mathbb{C}) = \underbrace{H^0(C,\Omega^1)}_{H^{1,0}(C)} \oplus \underbrace{H^1(C,\mathcal{O})}_{H^{0,1}(C)} \qquad H^{01} = \overline{H^{10}}$$

$$\omega_1, \dots, \omega_g \qquad \overline{\omega_1}, \dots, \overline{\omega_g}$$

Choose ω_j's s.t. $\int_{\beta_k} \omega_j = \delta_{jk}$

Define $\Omega \triangleq \left(\int_{\alpha_k} \omega_j \right)_{g\times g}$ period matrix

- $\Omega = {}^t\Omega \qquad (\because \int_C \omega_j \wedge \omega_k = 0)$
- $\operatorname{Im}\Omega > 0 \qquad (\because i\int \omega_j \wedge \overline{\omega_j} > 0)$

Given $p_0 \in C \rightsquigarrow \mu : C \longrightarrow \operatorname{Pic}^0(C)$

$p \longmapsto \mathcal{O}_C(p - p_0)$.

§ Abel theorem

Given $p_0 \in C \rightsquigarrow$

$$\kappa : C \longrightarrow \frac{H_1(C,\mathbb{R})}{H_1(C,\mathbb{Z})} =: \mathrm{Alb}(C), \qquad \kappa(p) = \int_{p_0}^{p}$$

Poincaré duality $H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z}) \xrightarrow{\cup} \mathbb{Z}$ perfect pairing.

$\rightsquigarrow \quad \Phi : H_1(C,\mathbb{Z}) \xrightarrow{\cong} H^1(C,\mathbb{Z})$

$$\rightsquigarrow \quad \Phi : \underbrace{\frac{H_1(C,\mathbb{R})}{H_1(C,\mathbb{Z})}}_{\mathrm{Alb}(C)} \xrightarrow{\cong} \underbrace{\frac{H^1(C,\mathbb{R})}{H^1(C,\mathbb{Z})}}_{\operatorname{Pic}^0(C)} \hookrightarrow H^1(C,\mathbb{C}) \to H^1(C,\mathcal{O})$$

Abel Thm : Given $p_0 \in C$

$$\begin{tikzcd} & \mathrm{Alb}(C) \arrow[d, "\Phi"] \\ C \arrow[ur, "\mu"] \arrow[r, "\kappa"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Pic}^0(C) =: J(C) \end{tikzcd}$$ Jacobian

$$\begin{tikzcd} & \int_{p_0}^{p} \arrow[d, dashed] \\ p \arrow[ur, mapsto] \arrow[r, mapsto] & \mathcal{O}_C(p-p_0) \end{tikzcd}$$

- $\sum \int_{q_r}^{p_r} = 0 \in \mathrm{Alb}(C) \Rightarrow \exists f: C \to \mathbb{CP}^1$ w/ $(f) = \sum p_r - \sum q_r$.

$$\kappa_r : \underbrace{C \times \cdots \times C}_{r\text{-times}} / S_r =: S^r C \longrightarrow J(C)$$

$$(p_1, \dots, p_r) \longmapsto \mathcal{O}_C\left(\sum_{j=1}^{r} p_j - r p_0\right)$$

$$\kappa_g : S^g C \longrightarrow J(C)$$ both $\dim_{\mathbb{C}} = g$

Claim. $\deg \kappa_g = 1$. Jacobi inversion thm.

i.e. $\forall \deg L = g \Rightarrow H^0(C, L) \neq 0$

generic such $L \Rightarrow H^0(C, L) \simeq \mathbb{C}$

Pf: $[\kappa(C)] = \sum_j \alpha_j \times \beta_j \in H_2(J(C), \mathbb{Z})$

↑Pontryagin product

$\rightsquigarrow$ $\kappa_g : H_{2g}(S^g C) \longrightarrow H_{2g}(J(C))$

$$\kappa_{g*}([S^g C]) = (\sum \alpha_j \times \beta_j)^g / g!$$
$$= \alpha_1 \times \beta_1 \times \dots \times \alpha_g \times \beta_g = \text{gen. of } H_{2g}(J(C), \mathbb{Z}).$$

$\Rightarrow$ deg 1 map.

§ Theta functions.

$\delta = [\delta_1, \dots, \delta_g]^t$, $\varepsilon = [\varepsilon_1, \dots, \varepsilon_g]^t$ w/ $\delta_j, \varepsilon_j = 0$ or 1

$$\vartheta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(\underset{\mathbb{C}^g}{\underset{\cap}{u}} ; \Omega) := \sum_{m \in \mathbb{Z}^g} e^{\pi i \left[(m+\frac{\delta}{2})^t \Omega (m+\frac{\delta}{2}) + 2(m+\frac{\delta}{2})(u+\frac{\varepsilon}{2}) \right]}$$

- cgt. $(\because \operatorname{Im}\Omega > 0)$

$$\vartheta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(u, \Omega) \rightsquigarrow \begin{cases} u \mapsto u + E_j = \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} & \times\ e^{-\pi i \delta_j} \\ u \mapsto u + \Omega_j & \times\ e^{-\pi i (2u_j + \varepsilon_j + \omega_{jj})} \end{cases}$$

- $\vartheta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(-u, \Omega) = \dots = e^{\pi i \langle \delta, \varepsilon \rangle} \vartheta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(u, \Omega)$.

Hence,

$$\vartheta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(u, \Omega) \text{ is } \begin{cases} \text{even} \iff \langle \delta, \varepsilon \rangle \text{ even} \\ \text{odd} \iff \langle \delta, \varepsilon \rangle \text{ odd} \end{cases}$$

Write $\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} = \sum \delta_j \alpha_j + \varepsilon_j \beta_j \in H_1(C, \mathbb{Z}_2)$

$q : H_1(C, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ (P.D.) quadratic form

$q\left(\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\right) \equiv \langle \delta, \varepsilon \rangle \pmod 2$

Define. $\Theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} :=$ Zero set of $\vartheta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(u; \Omega) \subseteq J(C)$

<u>Claim.</u> $\# \Theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \cap C = g$.

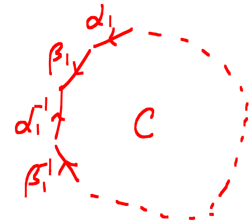
<u>Claim.</u> (up to translat^n.) $J(C) \longrightarrow S^g C$

$e \longmapsto \left(\Theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} + e \right) \cdot C$

is inverse to $\kappa_g : S^g C \longrightarrow J(C)$ $\kappa_g = \mathcal{O}_C(\Sigma P_j - g P_0)$.

i.e. $\displaystyle\sum_{p\in(\Theta[{\delta \atop \varepsilon}]+e)\cdot C}^{g} p \;-\; \sum_{p\in \Theta[{\delta \atop \varepsilon}]+ C}^{g} p \overset{?}{=} e \quad \in J(C).$

Pf: Compute $\int d\log\vartheta$ along



and $\int u\, d\log\vartheta$

using $u|_{\alpha_j^{-1}} = u|_{\alpha_j} + E_j$ & $u|_{\beta_j^{-1}} = u|_{\beta_j} + \Omega_j$ #

§ Riemann Theorem.

$$\Theta[{0 \atop 0}] = \kappa_{g-1}(S^{g-1}C) + K_{p_0} \subseteq J(C)$$

where $\displaystyle\sum_{\substack{0=\vartheta[{0 \atop 0}](u(p)-e-K_{p_0}) \\ p\in C}} u(p) = e$ (determine K_{p_0})

(Indep. of e by previous claim.)

Pf: Pick $p_1, \dots\dots, p_g \in C$

s.t. $\dim H^0(\mathcal{O}_C(p_1+\dots+p_g)) = 1$ (generic)

$e := u(p_1) + \dots + u(p_g)$

& $\mathrm{Zero}(\vartheta(u(-) - e - K_{p_0})) \not\supseteq C$

By def^n of K_{p_0},

$\mathrm{Zero}(\vartheta(u(-) - e - K_{p_0})) = \{p_1, \dots, p_g\}$

$\Rightarrow \quad \vartheta(\underbrace{u(p_1) - e}_{-u(p_2)-\dots-u(p_g)} - K_{p_0}) = 0$

But $p_2 + \dots + p_g$ $(g-1)$-pt can vary generically in $S^{g-1}C$

$\Rightarrow \quad \kappa_{g-1}(S^{g-1}C) + K_{p_0} \subseteq \Theta[{0 \atop 0}].$

. $[\supseteq]$ is similar #

Recall $\dim H^0(C, \mathcal{O}(p_1+\dots+p_g)) \geq 1$.

Claim: $\underline{\qquad\qquad\qquad\text{"}\qquad\qquad\qquad} = 1$

$\Longleftrightarrow$ $\Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \sum_{j=1}^{g} p_j + K_{p_0} \not\supseteq \kappa(C)$

$\Longrightarrow$ $\left(\Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \sum_{j=1}^{g} p_j + K_{p_0}\right) \cap \kappa(C) = p_1+\dots+p_g$

Pf: If $\dim > 1$

$\forall\ q \in C\quad \exists\ q_2, \dots, q_g$

s.t. $u(q)+u(q_2)+\dots+u(q_g) = \sum_{j=1}^{g} u(p_j) =: e$

$\Rightarrow$ $\vartheta(\underbrace{u(q) - e}_{-\sum_1^{g-1} u(q_j)\ \in\ \kappa_{g-1}(S^{g-1}C)} - K_{p_0}) = 0$

$\left(\because \begin{array}{l}\text{Riemann thm.} \quad + \quad \vartheta : \text{even}\\ \Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = \kappa_{g-1}(S^{g-1}C) + K_{p_0}\end{array}\right)$

Hence $\Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \sum u(p_j) + K_{p_0} \supseteq \kappa(C)$.

Converse is similar. #

Claim: $\Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = \Theta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right] + I\frac{\varepsilon}{2} + \Omega\cdot\frac{\delta}{2}$

Pf: $g(u) := e^{-\pi i \sum u_j \delta_j} \times \dfrac{\vartheta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right](u\,;\Omega)}{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](u + I\frac{\varepsilon}{2} + \Omega\frac{\delta}{2})}$

$\left(u \mapsto \begin{array}{l} u+E_j \\ u+\Omega_j\end{array}\ \text{transf} \Rightarrow\right)$ $g : J(C) \longrightarrow \mathbb{P}^1$

$e \in J(C)$ generic, $\quad g(u+e)|_C : C \longrightarrow \mathbb{P}^1$

IF not const. $\Rightarrow$ pole set q_j's (i.e. zero of $\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$)
has $\dim H^0(\mathcal{O}_C(q_1+\dots+q_g)) = 1$
(by previous claim).

But $C \to \mathbb{P}^1 \Rightarrow q_1 + \dots + q_g$ move

i.e. $\dim H^0 \geq 2$ $(*)$

$\Rightarrow \vartheta(u+e)|_C$ is const. fu. for generic $e \in J(C)$

$\Rightarrow \vartheta(u) \equiv$ const. $\#$

$$J(C) \longrightarrow J(C) \longrightarrow \mathrm{Pic}^{g-1}(C)$$

$u \longmapsto u - K_{p_0}$ $\quad L \longmapsto L((g-1)p_0)$ $\quad \cup$ ← canon. def$^{\text{d}}$. w/o p_0.

i.e. $\Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \longleftrightarrow \Theta = K_{g-1}(S^{g-1}C)$

$$\underbrace{\left\{\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right]'s\right\}}_{H_1(C,\mathbb{Z}) \simeq \frac{H_1(C,\frac{1}{2}\mathbb{Z})}{H_1(C,\mathbb{Z})} \subseteq \frac{H_1(C,\mathbb{R})}{H_1(C,\mathbb{Z})} = J(C)} \xleftrightarrow{\text{claim.}} \Sigma = \{L : L^{\otimes 2} = K_C\}$$

↑ theta characteristic (spin structure)

§ Riemann Singularities Theorem

$L \in \Theta = \kappa(S^{g-1}C) \subset \mathrm{Pic}^{g-1}(C)$ (i.e. $H^0(C,L) \neq 0$)

- $L \in \Theta_{sg} \iff \dim H^0(C,L) > 1$

  (singularity set of Θ)

- $\mathrm{mult}_L \Theta = \dim H^0(C,L)$

Pf: $\dim H^0(C,L) > s$ $\qquad L = \mathcal{O}(p_1 + \dots + p_{g-1})$

Let $e = \sum_{j=1}^{g-1} u(p_j)$

$\forall\, q_1, \dots, q_s \in C$ $\quad \exists\, q_{s+1}, \dots, q_{g-1} \in C$

s.t. $e = \sum_{j=1}^{g-1} u(q_j)$

$$\Rightarrow \vartheta\Big(\underbrace{\sum_1^s u(p_j)}\ - \sum_1^s u(q_j) \underline{- e} - K_{p_0}\Big)$$

$$= \vartheta\Big(-\sum_{s+1}^{g-1} u(p_j) - \sum_1^s u(q_j) - K_{p_0}\Big)$$

$$= \vartheta\Big(\underbrace{+\sum_{s+1}^{g-1} u(p_j) + \sum_1^s u(q_j) + K_{p_0}}_{\in K_{g-1}(S^{g-1}C) + K_p \,=\, \Theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \text{ (Riemann Thm).}}\Big) \qquad (\because \text{ even fu.})$$

$$= 0$$

$$\Rightarrow \quad \vartheta\Big(K_s(S^sC) - K_s(S^sC) - \sum_{j=1}^{g-1} u(p_j) - K_{p_0}\Big) = 0$$

(Converse also true.)

For simplicity, $s = 1$.

$$\Rightarrow \forall\, p \in C \qquad \underbrace{\lim_{q \to p} \frac{\vartheta(u(q) - u(p) - e - K_{p_0})}{z(q) - z(p)}}_{\sum_j \frac{\partial \vartheta}{\partial u_j}(-e - K_{p_0}) \cdot \frac{\partial u_j}{\partial z}(p)} = 0$$

z: local cpx. coord. around p

Vary $p \in C$ $\Rightarrow \quad \dfrac{\partial \vartheta}{\partial u_j}(-e - K_{p_0}) = 0 \qquad \forall\, j$

i.e. $\quad \mathrm{mult}\ \vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](u, \Omega) > 1$ at $L \otimes L_{p_0}^{g-1} + K_{p_0}$.

Similar for bigger s. $\qquad [\Rightarrow]$ ✓ $\qquad$ #

Remark: C general (in fact non-hyperelliptic)

$$\Rightarrow \quad \dim \Theta_{sg} = g - 4$$

reason: $C \in \mathcal{M}^{3g-3}$, $\ L \in \Theta_{sg}$ (i.e. $\dim H^0(L) > 1$) $\rightsquigarrow$ $C \xrightarrow[\text{cover}]{(g-1)\text{-sheet}} \mathbb{CP}^1$

branch pt. $= 4g - 4$.

moving branch pts. $\rightsquigarrow$ $4g - 4 - \underbrace{\dim \mathrm{Aut}\,\mathbb{P}^1}_{3}$

$$\Rightarrow \quad 3g - 3 + \dim \Theta_{sg} = 4g - 4 - 3$$

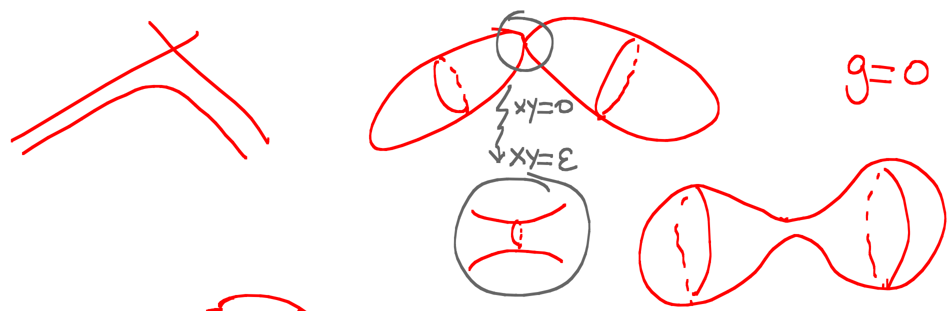
$$\Rightarrow \quad \dim \Theta_{sg} = g - 4.$$

Chapter 5. Quartic & Quintic.

$$C = \{ F(X_0, X_1, X_2) = 0 \} \subset \mathbb{CP}^2$$

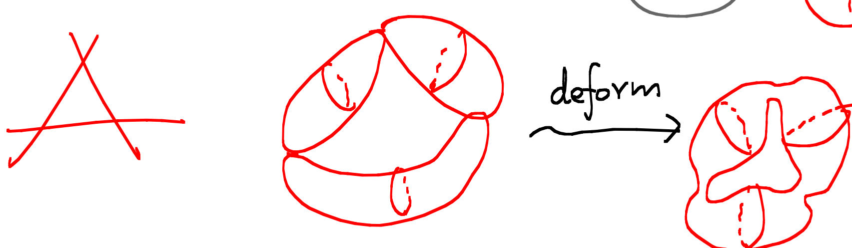
$$\deg F = d \implies \underline{\text{genus}(C) = \frac{(d-1)(d-2)}{2}}$$

Eg. $d=2$



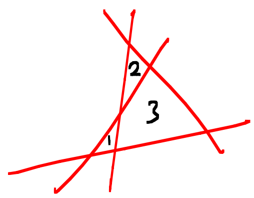
$g=0$

$d=3$



$g=1$

$d=4$



$\implies \quad g = 3 \qquad$ etc.

$$\kappa : C \longrightarrow J(C)$$

$\forall\ p \in C$, $d\kappa(p) : T_p C \longrightarrow T_{\kappa(p)} J(C) \simeq \underbrace{T_0 J(C)}$

$\rightsquigarrow$ 1 dim. subsp. in $\underbrace{H^1(\mathcal{O}_C)}_{H^{0,1}(C)}$

$\rightsquigarrow$ Gauss map $\quad \mathcal{G} : C \longrightarrow \mathbb{P}(H^{0,1}(C)) \simeq \mathbb{P}^{g-1}$

$$\qquad \qquad H^{0,1}(C) \,\|\, H^{1,0}(C)^*$$

Exercise: $\quad \mathcal{G}(p) = \{ \omega \in H^{1,0}(C) \mid \omega(p) = 0 \}$

i.e. Gauss map = canonical map.

$$\Phi_K : C \longrightarrow |K_C| \cong \mathbb{P}^{g-1}.$$

Therefore, if φ NOT injective

$\Longrightarrow$ $\exists\ p \neq q$ s.t. $\omega(p)=0$ iff $\omega(q)=0$ $\forall \omega \in \Omega^1(C)$.

$\xrightarrow{\text{R.R.}}$ $H^0(C\ \mathcal{O}(p+q)) \simeq \mathbb{C}^2$ i.e. hyperelliptic $C \xrightarrow{2:1} \mathbb{P}^1$

Conclusion, C non-hyperelliptic

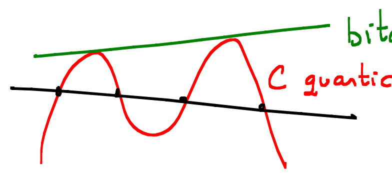
$\Longrightarrow$ $\varphi : C \underset{\text{emb.}}{\hookrightarrow} \mathbb{P}^{g-1}$ of deg. $2g-2$.

§ Quartics

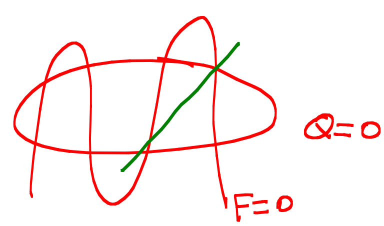
Eg. $g=3$, non-hyperelliptic $\Longleftrightarrow$ smooth quartic in $\mathbb{P}^2$.

canonical divisors are hyperplane sections.

28 bitangents to C



(1) $C_\varepsilon = \{\varepsilon F + Q^2 = 0\}$



$p, q \in F \cap Q = 8$ pts.

$\overline{pq}$ perturbs to bitangent to C_ε

$\Rightarrow$ $\# = \binom{8}{2} = 28$.

(2) L bitangent i.e. $L \cdot C = 2p + 2q \in |K_C|$

$\Rightarrow$ $\mathcal{O}(p+q)$ is theta characteristic

w/ $\dim H^0(\mathcal{O}(p+q)) \geq 1$ ($\gneq 1 \Rightarrow$ hyperelliptic ($\not\to$))

i.e. <u>odd</u> theta char.

$\#$ odd theta char. $= 2^{g-1}\cdot(2^g-1) = 2^2\cdot(2^3-1) = 28$

$\#$

$g=3$ hyperelliptic $\Longrightarrow$
$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \mathbb{P}(H^{1,0*}) \simeq \mathbb{P}^2 \\ {\scriptstyle 2:1}\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

$C: \ y^2 = f(x) \quad \deg f = 7$ ($\because$ # branch pt. $=2g+2=8$)

$$H^{1,0}(C) = \mathbb{C}\left\langle \frac{dx}{y}, \frac{x\,dx}{y}, \frac{x^2\,dx}{y} \right\rangle$$

$$\varphi = [1, x, x^2]$$

$\Rightarrow$ $\varphi(C) \subset \mathbb{P}^2$ is a conic
$\quad \| $
$\{Q=0\}$

Nearby $g=3$ curves : $\{\varepsilon F + Q^2 = 0\} \subset \mathbb{P}^2$.

§ Quintic.

$C \subset \mathbb{P}^2 \quad \deg C = 5$

$$\Longrightarrow \quad g = \frac{(d-1)(d-2)}{2} = 6.$$

- $\{\text{quintic}\}/\cong \quad \subseteq \quad \mathcal{M}_{g=6}$

dim: $\binom{5+2}{2} - \dim GL(3) = 12 \ \lneq \ 3g-3 = 15$

i.e. most $g=6$ curves are not quintic in $\mathbb{P}^2$

- $K_C = \mathcal{O}_{\mathbb{P}^2}(2)|_C \qquad (\deg K_C = 2g-2 = 10 = 2 \times \deg C)$

canon. div. $\longleftrightarrow$ $C \cap$ conic in $\mathbb{P}^2$

- $\mathcal{O}_{\mathbb{P}^2}(1)|_C$ distinguished theta char.

$\dim H^0(C, \mathcal{O}(1)) = 3 \qquad (0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_C(1) \to 0)$

i.e. $\Theta \subseteq J(C)$ has a triple pt.
($\because$ Riemann sing. thm)

- Generically,

  plane quintic $\longleftrightarrow$ $g=5$ curve same dim. $(3g-3=12)$

$g(B)=5$ generic

$\Longrightarrow$ $B = Q_0 \cap Q_1 \cap Q_2 \subseteq \mathbb{CP}^4$

c.i. of 3 quadrics

$$= \bigcap_{[\lambda,\mu,\nu]\in\mathbb{P}^2} \{\lambda Q_0 \cap \mu Q_1 \cap \nu Q_2 = 0\}$$

$\emptyset$

Singular if $\underbrace{\det(\lambda \overset{5\times 5 \text{ matrix}}{Q_0} + \mu Q_1 + \nu Q_2)}_{\deg 5} = 0$

$\rightsquigarrow$ quintic in $\mathbb{P}^2$.

In fact $B \subseteq \mathbb{CP}^4 = |K_B|$ canon. emb.

$\forall\ u_0 \in \Theta_{sg}$

Riemann sing. thm $\Longrightarrow$ $B \subset \overset{\text{quadric cone}}{Q} \subset \mathbb{P}^4$

$\rightsquigarrow$ $\Theta_{sg} \longrightarrow C$ unbranched double cover.

Chapter 6. Schottky relation

§ Prym

$\pi : \tilde{C} \xrightarrow{2:1} C$ unramified double cover

$\leftrightarrow \quad 1 \to \underbrace{\Gamma}_{\pi_1(\tilde{C})} \longrightarrow \pi_1(C) \longrightarrow \mathbb{Z}_2 \to 0$

$\leftrightarrow$ index 2 $\underbrace{S}_{H_1(\tilde{C},\mathbb{Z}_2)} \leq H_1(C,\mathbb{Z}_2)$

$\leftrightarrow \quad \beta_0 \in H_1(C,\mathbb{Z}) \setminus 0 \qquad (S = \gamma^{\perp})$

- $g(C) = g+1 \Rightarrow g(\tilde{C}) = 2g+1 \quad (\because \chi(\tilde{C}) = 2\chi(C))$
- $\begin{array}{c} \tilde{C} \circlearrowleft \iota \\ \pi \downarrow \\ C \end{array}$ involution $\Rightarrow$ $\iota_* : H_1(\tilde{C},\mathbb{Z}) \circlearrowleft \qquad \iota^* : H^{1,0}(\tilde{C}) \circlearrowleft$

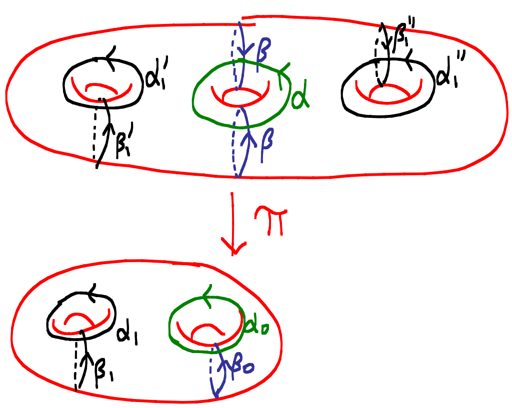
  $\iota_*^2 = 1 \qquad \iota^{*2} = 1$

  eigenvalue $= -1$ & $+1 (\sim H(C))$

$$\mathrm{Prym}(\tilde{C}/C) := \mathrm{Jac}(C)^- = \frac{H^{1,0}(\tilde{C})^{-\,*}}{H_1(\tilde{C},\mathbb{Z})^-}$$

- $\beta_0 \subset C$ simple closed curve

  $\rightsquigarrow$ double cover $\pi : \tilde{C} \xrightarrow{2:1} C \Rightarrow \pi^{-1}\beta_0$ disconnect $\tilde{C}$



$\alpha \quad \beta \quad \alpha_1' \ \beta_1' \ \cdots \ \alpha_g' \ \beta_g'$

$ \quad \alpha_1'' \ \beta_1'' \ \cdots \ \alpha_g'' \ \beta_g''$

$\alpha_0 \quad \beta_0 \quad \alpha_1 \ \beta_1 \ \cdots \ \alpha_g \ \beta_g$

$$\pi_* \quad \begin{array}{cccc} \alpha & \beta & \alpha_j' \ \alpha_j'' & \beta_j' \ \beta_j'' \\ \downarrow & \downarrow & \searrow \swarrow & \searrow \swarrow \\ 2\alpha_0 & \beta_0 & \alpha_j & \beta_j \end{array}$$

$$H_1(\tilde{C}, \mathbb{Z})^- = \mathbb{Z}\langle \alpha_j' - \alpha_j'', \beta_j' - \beta_j'' \rangle_{j=1}^{g} \quad (\dim = g)$$

Intersection pairing: $2\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$

$$H^{1,0}(\tilde{C})^- = \mathbb{C}\langle \underset{\text{choose}}{\psi_1, \ldots, \psi_g} \rangle, \quad \text{s.t.} \quad \int_{\beta_j'-\beta_j''} \psi_k = \delta_{jk}$$

$$\Gamma \overset{\Delta}{=} \Big(-\int_{\alpha_j'-\alpha_j''} \psi_k\Big)_{g\times g} = {}^t\Gamma \ \& \ \mathrm{Im}\,\Gamma > 0$$

$$\eta\Big[\begin{smallmatrix} \delta \\ \varepsilon \end{smallmatrix}\Big](v, \Gamma) \overset{\Delta}{=} \sum_{m\in\mathbb{Z}^g} e^{\pi i\left[{}^t(m+\frac{\delta}{2})\Gamma(m+\frac{\delta}{2}) + 2\,{}^t(m+\frac{\delta}{2})(v+\frac{\varepsilon}{2})\right]}$$

Note: $g(B) = 5 \rightsquigarrow \pi : \Theta_{sg}(B) \xrightarrow[2:1]{\text{unramified}} \underset{\text{quintic}}{C} \subset \mathbb{P}^2$

$\Rightarrow \mathrm{Prym}(\pi) = \mathrm{Jac}(B)$

- Schottky problem. $C_g \rightsquigarrow J(C) = A^g$ (pr. pol. Abelian variety) $= \mathbb{C}^g/\mathbb{Z}^g + \Sigma\mathbb{Z}A_j$

$$\{C\}^{3g-3} \hookrightarrow \{A\}^{g(g+1)/2}$$

$g(C) \geq 4 \Rightarrow$ NOT surjective. Image = ?

eg. $g(C) = 4$ ($9 < 10$) $\exists$ 1 (Schottky) relation.

Find it !

Idea: $A = J(C)$. Can take (many) $\pi : \tilde{C} \xrightarrow[\text{unram.}]{2:1} C$

$\rightsquigarrow \mathrm{Prym}(\pi)$ st. 'Riemann identity'.

$\vartheta \underset{\text{proportional}}{\sim} \eta$ (Schottky - Jung)

$\rightsquigarrow$ new relation.

§ Riemann Theta Relation.

Lattice (Preliminary)

$$L_1 = \mathbb{Z}^4 = \mathbb{Z}\langle e_1, e_2, e_3, e_4\rangle \leq (\mathbb{R}^4, \langle\ \rangle)$$

$$L_2 \triangleq \mathbb{Z}\left\langle (e_j + e_k)\text{'s}, \frac{e_1+e_2+e_3+e_4}{2}\right\rangle$$

o.n. basis : $\begin{cases} f_1 = (e_1 + e_2 + e_3 + e_4)/2 \\ f_2 = (\ +\ -\ -\ )/2 \\ f_3 = (\ -\ +\ -\ )/2 \\ f_4 = (\ -\ -\ +\ )/2 \end{cases}$

$$M : L_1 \xrightarrow{\cong} L_2 \quad \text{as abstract lattices}$$
$$e_j \mapsto f_j$$

$$M : \mathbb{R}^4 \circlearrowleft \quad \text{s.t.} \quad M^2 = 1$$

Note:

$$\begin{tikzcd} & L_1 \arrow[dr, phantom, "\leq\, f_1"] & \\ L_1 \cap L_2 \arrow[ur, phantom, "\leq"] \arrow[dr, phantom, "\leq"] & & L_1 + L_2 =: L \\ & L_2 \arrow[ur, phantom, "\leq\, e_1"] & \end{tikzcd}$$

all index 2.

- $$2\sum_{m\in L_1} f(m) = \underbrace{\sum_{m\in L_1} f(m) + \sum_{m\in L_1+f_1} f(m)}_{\sum_{m\in L} f(m)} \underbrace{+\sum_{m\in L_1} f(m) - \sum_{m\in L_1+f_1} f(m)}_{\sum_{m\in L} e^{\pi i\cdot 2(m\cdot e_1)} f(m)}$$

$$\left(\because m \in L = L_1 \sqcup (L_1+f_1) \Rightarrow m\cdot e_1 \in \begin{cases} \mathbb{Z} & \text{if } m \in L_1 \\ \mathbb{Z}+\frac{1}{2} & \text{if } m \in L_1+f_1 \end{cases}\right)$$

- Say $f(m) = e^{2\pi i(a(m+g)^2 + b(m+g))}$, then 2^{nd} term

$$\sum_{m\in L} e^{\pi i\cdot 2(m\cdot e_1)} f(m) = \sum_{m\in L} e^{2\pi i(m+g-g)\cdot e_1} e^{2\pi i[a(m+g)^2 + b(m+g)]}$$

$$= \sum_{m\in L} e^{2\pi i(-g\cdot e_1)} e^{2\pi i[a(m+g)^2 + (b+e_1)(m+g)]}$$

Say $g=1$ case, $A_{1\times 1}=\{\tau\}$, $\operatorname{Im} A>0$

$g_j, h_j \in \mathbb{R}^g$

$$\vartheta\begin{bmatrix} g_1 \\ h_1 \end{bmatrix}(u_1,A)\,\vartheta\begin{bmatrix} g_2 \\ h_2 \end{bmatrix}(u_2,A)\,\vartheta\begin{bmatrix} g_3 \\ h_3 \end{bmatrix}(u_3,A)\,\vartheta\begin{bmatrix} g_4 \\ h_4 \end{bmatrix}(u_4,A) = \prod_{j=1}^{4}\vartheta\begin{bmatrix} g_j \\ h_j \end{bmatrix}(u_j,A)$$

$$= \sum_{m_j\in\mathbb{Z}} e^{2\pi i\left[\sum_{j=1}^{4} \underbrace{\frac{1}{2}\left(m_j+\frac{1}{2}g_j\right)^2 A}_{} + \left(m_j+\frac{1}{2}g_j\right)\left(u_j+\frac{1}{2}h_j\right)\right]}$$

$$\begin{pmatrix} m_1+g_1/2 & m_2+g_2/2 & m_3+g_3/2 & m_4+g_4/2 \end{pmatrix} \underbrace{\begin{pmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{pmatrix}}_{\mathcal{A}} \begin{pmatrix} \underbrace{m_1}_{} + g_1/2 \\ m_2 + g_2/2 \\ m_3 + g_3/2 \\ \underbrace{m_4}_{m\in L_1} + g_4/2 \end{pmatrix}$$

$$= \sum_{m\in L_1} e^{2\pi i\left[\frac{1}{2}\,{}^t\!\left(m+\frac{1}{2}g\right)\mathcal{A}\left(m+\frac{1}{2}g\right) + {}^t\!\left(m+\frac{1}{2}g\right)\left(u+\frac{1}{2}h\right)\right]}$$

$$= \frac{1}{2}\sum_{m\in L}\left(-\,''\,-\right) + \frac{1}{2}\sum_{m\in L} e^{2\pi i(m\cdot e_1)}\left(-\,''\,-\right)$$

$$\left(L = L_2 \amalg (L_2+e_1)\right)$$

$$= \frac{1}{2}\sum_{m\in L_2}\left[\begin{aligned} & e^{2\pi i\left[\frac{1}{2}\,{}^t(m+\frac{1}{2}g)\mathcal{A}(m+\frac{1}{2}g) + {}^t(m+\frac{1}{2}g)(u+\frac{1}{2}h)\right]} \\ +\; & e^{2\pi i\left[\frac{1}{2}\,{}^t(m+e_1+\frac{1}{2}g)\mathcal{A}(m+e_1+\frac{1}{2}g) + {}^t(m+e_1+\frac{1}{2}g)(u+\frac{1}{2}h)\right]} \\ +\; & e^{2\pi i\frac{-g\cdot e_1}{2}}\, e^{2\pi i\left[\frac{1}{2}\,{}^t(m+\frac{1}{2}g)\mathcal{A}(m+\frac{1}{2}g) + {}^t(m+\frac{1}{2}g)(u+e_1+\frac{1}{2}h)\right]} \\ +\; & e^{2\pi i\frac{-g\cdot e_1}{2}}\, e^{2\pi i\left[\frac{1}{2}\,{}^t(m+e_1+\frac{1}{2}g)\mathcal{A}(m+e_1+\frac{1}{2}g) + {}^t(m+e_1+\frac{1}{2}g)(u+e_1+\frac{1}{2}h)\right]} \end{aligned}\right]$$

$$= \frac{1}{2}\sum_{m\in L_1}\left[\begin{aligned} & e^{2\pi i\left[\frac{1}{2}\,{}^t(m+\frac{1}{2}Mg)\mathcal{A}(m+\frac{1}{2}Mg) + {}^t(m+\frac{1}{2}Mg)(Mu+\frac{1}{2}Mh)\right]} \\ +\; & e^{2\pi i\left[\frac{1}{2}\,{}^t(m+\cancel{e_1}f_1+\frac{1}{2}Mg)\mathcal{A}(m+\cancel{e_1}f_1+\frac{1}{2}Mg) + {}^t(m+\cancel{e_1}f_1+\frac{1}{2}Mg)(Mu+\frac{1}{2}Mh)\right]} \\ +\; & e^{2\pi i\frac{-g\cdot e_1}{2}}\, e^{2\pi i\left[\frac{1}{2}\,{}^t(m+\frac{1}{2}Mg)\mathcal{A}(m+\frac{1}{2}Mg) + {}^t(m+\frac{1}{2}Mg)(Mu+\cancel{e_1}f_1+\frac{1}{2}Mh)\right]} \\ +\; & e^{2\pi i\frac{-g\cdot e_1}{2}}\, e^{2\pi i\left[\frac{1}{2}\,{}^t(m+\cancel{e_1}f_1+\frac{1}{2}Mg)\mathcal{A}(m+\cancel{e_1}f_1+\frac{1}{2}Mg) + {}^t(m+\cancel{e_1}f_1+\frac{1}{2}Mg)(Mu+\cancel{e_1}f_1+\frac{1}{2}Mh)\right]} \end{aligned}\right]$$

$$\Rightarrow \prod_{j=1}^{4} \vartheta\begin{bmatrix} g_j \\ h_j \end{bmatrix}(u_j; A) = \frac{1}{2} \sum_{\left[\begin{smallmatrix}\alpha' \\ \alpha''\end{smallmatrix}\right] \in \mathbb{Z}_2^2} e^{-\pi i \alpha''\cdot g_1} \prod_{j=1}^{4} \vartheta\begin{bmatrix} Mg_j + \alpha' \\ Mh_j + \alpha'' \end{bmatrix}(Mu_j, A)$$

(Same for ANY g w/ $M_{4g \times 4g}$, etc.)

Special case: $\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4 = \frac{1}{2}\left(\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4 + \vartheta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^4 + \vartheta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^4\right)$

(i.e. Riemann theta relation).

§ Use $L_1 = \mathbb{Z}^2 \overset{\frac{e_1+e_2}{2}}{<} L_2 = \mathbb{Z}\langle \frac{e_1+e_2}{2}, \frac{e_1-e_2}{2} \rangle = M^{-1}L$ w/ $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ s.t. $M^2 = -1$

Similarly $\Rightarrow$

$$\vartheta\begin{bmatrix} g_1 \\ h_1 \end{bmatrix}(u_1, A)\, \vartheta\begin{bmatrix} g_2 \\ h_2 \end{bmatrix}(u_2, A) = \sum_{\alpha'=0,1} \vartheta\begin{bmatrix} M^{-1}g_1 \\ M\, h_1 \end{bmatrix}(Mu_1, 2A)\; \vartheta\begin{bmatrix} M^{-1}g_2 + \alpha' \\ M\, h_2 \end{bmatrix}(Mu_2, 2A)$$

§ Proportionality. $\pi: \tilde{C} \xrightarrow[2:1]{\text{unramified}} C$

$$\Gamma = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1g} \\ \vdots & & \vdots \\ \tau_{g1} & \cdots & \tau_{gg} \end{pmatrix}_{g^2} \qquad \tilde{\Omega} = \left(\begin{array}{c|c|c} 2\omega_{00} & \omega_{01} \cdots \omega_{0g} & \omega_{01} \cdots \omega_{0g} \\ \hline \begin{matrix}\omega_{10} \\ \vdots \\ \omega_{g0}\end{matrix} & \dfrac{\omega_{ij}+\tau_{ij}}{2} & \dfrac{\omega_{ij}-\tau_{ij}}{2} \\ \hline \begin{matrix}\omega_{10} \\ \vdots \\ \omega_{g0}\end{matrix} & \dfrac{\omega_{ij}-\tau_{ij}}{2} & \dfrac{\omega_{ij}+\tau_{ij}}{2} \end{array}\right)\begin{matrix}1\\g\\g\end{matrix} \qquad \Omega = \begin{pmatrix} \omega_{00} & \cdots & \omega_{0g} \\ \vdots & \ddots & \vdots \\ \omega_{g0} & \cdots & \omega_{gg} \end{pmatrix}_{(g+1)^2}$$

for Prym(π) $\qquad$ Jac($\tilde{C}$) $\qquad$ Jac(C)

$$M\tilde{\Omega}M = \begin{bmatrix} 2\Omega & \\ & 2\Gamma \end{bmatrix}\begin{matrix} g+1 \\ g \end{matrix} \text{ where } M = \begin{pmatrix} 1 & & \\ & I & I \\ & I & -I \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & & \\ & I/2 & I/2 \\ & I/2 & -I/2 \end{pmatrix}$$

Define

$$\vartheta\begin{bmatrix} e' & \varepsilon' & \varepsilon' \\ e'' & \varepsilon'' & \varepsilon'' \end{bmatrix}(w, \tilde{\Omega}) = \sum_{\substack{(m_0, m, n) \in \\ \mathbb{Z}\times\mathbb{Z}^g\times\mathbb{Z}^g}} e^{2\pi i\left[\frac{1}{2}{}^t\!\begin{pmatrix} m_0 + e'/2 \\ m + \varepsilon'/2 \\ n + \varepsilon'/2 \end{pmatrix}\tilde{\Omega}\begin{pmatrix} m_0 + e'/2 \\ m + \varepsilon'/2 \\ n + \varepsilon'/2 \end{pmatrix} + {}^t\!\begin{pmatrix} m_0 + e'/2 \\ m + \varepsilon'/2 \\ n + \varepsilon'/2 \end{pmatrix}\left(w + \begin{pmatrix} e'' \\ \varepsilon'' \\ \varepsilon'' \end{pmatrix}\right)\right]}$$

Consider $L_2 = M^{-1}(\mathbb{Z}^{2g+1}) \overset{\text{index } 2}{\supseteq} \mathbb{Z}^{2g+1}$)

$$\overset{(\star)}{=} \sum_{\alpha' \in \mathbb{Z}_2^g} (-1)^{(\varepsilon' + \alpha')\cdot \varepsilon''}\, \vartheta\begin{bmatrix} e' & \varepsilon' + \alpha' \\ e'' & 0 \end{bmatrix}_{g+1}(u, 2\Omega)\;\; \eta\begin{bmatrix} \alpha' \\ 0 \end{bmatrix}_g (v, 2\Gamma)$$

$u = [z_0, z_1 + z_{g+1}, \ldots, z_g + z_{2g}]$

$v = [z_1 - z_{g+1}, \ldots, z_g - z_{2g}]$

$$\begin{array}{c}\tilde{C}\\ \pi\downarrow\\ C\end{array}$$

choose theta char. L/C odd w/ $\pi^*L/\tilde{C}$ even.

(wrt our bases: $\Updownarrow$ $e'e''+\varepsilon'\cdot\varepsilon''\in 2\mathbb{Z}+1$ $\qquad \Uparrow$ $e'=0$ $\qquad$ [Topological, easy to verify.]

$$\Rightarrow\ h^0(C,L)\neq 0\ (\because \text{odd}) \Rightarrow h^0(\tilde{C},\pi^*L)\neq 0 \quad (\because \text{pullback})$$

$$\Rightarrow\ h^0(\tilde{C},\pi^*L)>1 \qquad \pi^*L \longleftrightarrow \left[\begin{smallmatrix}0 & \varepsilon' & \varepsilon'\\ 1 & \varepsilon'' & \varepsilon''\end{smallmatrix}\right] \text{ w/ } \varepsilon'\cdot\varepsilon''\equiv 1\ (2)$$

$$\Rightarrow\ 0=\vartheta\left[\begin{smallmatrix}0 & \varepsilon' & \varepsilon'\\ 1 & \varepsilon'' & \varepsilon''\end{smallmatrix}\right]\left(\int_p^q dw\ ;\ \tilde{\Omega}\right) \qquad (\text{for } \tilde{C})$$

$$\left(\because 1<\dim H^0(\tilde{C},\underbrace{\pi^*L}_{\tilde{L}}) = \mathrm{multi}_{\tilde{L}}\,\Theta = \mathrm{multi}_k\,\vartheta\left[\begin{smallmatrix}\tilde{\delta}\\ \tilde{\varepsilon}\end{smallmatrix}\right](w,\tilde{\Omega})\right.$$

$$k=\tilde{L}\otimes L_{p_0}^{-(\tilde{g}-1)}+K_{p_0}+I\,\tilde{\varepsilon}/2+\tilde{\Omega}\,\tilde{\delta}/2$$

$$\Updownarrow$$

$$\vartheta\left(\kappa_1(\tilde{C})-\kappa(\tilde{C})-\Sigma\,\mathcal{U}(p_k)-K_{p_0}\right)=0 \qquad \pi^*L=\mathcal{O}(p_1+\dots+p_{\tilde{g}-1})\Big)$$

$$\overset{\text{by }(\star)}{\Rightarrow}\ 0=\sum_{\alpha'}(-1)^{(\varepsilon'+\alpha')\cdot\varepsilon''}\vartheta\left[\begin{smallmatrix}0 & \varepsilon'+\alpha'\\ 1 & 0\end{smallmatrix}\right]\left(\int_p^q du;2\Omega\right)\ \eta\left[\begin{smallmatrix}\alpha'\\ 0\end{smallmatrix}\right]\left(\int_p^q dv;2\Gamma\right)$$

(for C) $\qquad$ (for $\mathrm{Prym}(\tilde{C}/C)$)

Fix any ε', sum over ε'' w/ $\varepsilon'\cdot\varepsilon''\equiv 1\ (2)$

$$\left(\text{Use } \sum_{\beta\cdot\varepsilon_1\equiv 0}(-1)^{(\varepsilon'+\alpha')\cdot(\varepsilon''+\beta)} = \begin{cases}2^{g-1} & \alpha'=\varepsilon'\\ -2^{g-1} & \alpha'=0\\ 0 & \text{otherwise}\end{cases}\right.$$

$$\Rightarrow\ \vartheta\left[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right]\left(\int_p^q du,2\Omega\right)\ \eta\left[\begin{smallmatrix}\varepsilon'\\ 0\end{smallmatrix}\right]\left(\int_p^q dv,2\Gamma\right) \qquad \forall\ \varepsilon'\neq 0$$

$$=\vartheta\left[\begin{smallmatrix}0 & \varepsilon'\\ 1 & 0\end{smallmatrix}\right]\left(\int_p^q du,2\Omega\right)\ \eta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\left(\int_p^q dv,2\Gamma\right)$$

$$\underset{\forall p,q\in\tilde{C}}{\Rightarrow}\ \left(\vartheta\left[\begin{smallmatrix}0 & \alpha'\\ 1 & 0\end{smallmatrix}\right]\left(\int_p^q du,\underline{2\Omega}\right)\right)_{\alpha'\in\mathbb{Z}_2^g} \overset{\text{proportional}}{\sim} \left(\eta\left[\begin{smallmatrix}\alpha'\\ 0\end{smallmatrix}\right]\left(\int_p^q dv\ ;\underline{2\Gamma}\right)\right)_{\alpha'\in\mathbb{Z}_2^g}$$

- Need to switch $\begin{smallmatrix}2\Omega\\ 2\Gamma\end{smallmatrix}$ back to $\begin{smallmatrix}\Omega\\ \Gamma\end{smallmatrix}$. (studied before)

§ Schottky - Jung Proportionality Theorem.

$$\left(\vartheta\left(\left[\begin{smallmatrix}0 & \varepsilon' \\ 1 & \varepsilon''\end{smallmatrix}\right]\right) \vartheta\left[\begin{smallmatrix}0 & \varepsilon' \\ 0 & \varepsilon''\end{smallmatrix}\right]\left(\textstyle\int du\right) + \vartheta\left(\left[\begin{smallmatrix}0 & \varepsilon' \\ 0 & \varepsilon''\end{smallmatrix}\right]\right) \vartheta\left[\begin{smallmatrix}0 & \varepsilon' \\ 1 & \varepsilon''\end{smallmatrix}\right]\left(\textstyle\int du\right) \right)_{\left(\begin{smallmatrix}\varepsilon' \\ \varepsilon''\end{smallmatrix}\right)} \quad (\text{for } C)$$

$$\underbrace{\ \ }_{\text{proportional}} \left(\eta\left(\left[\begin{smallmatrix}\varepsilon' \\ \varepsilon''\end{smallmatrix}\right]\right) \ \eta\left[\begin{smallmatrix}\varepsilon' \\ \varepsilon''\end{smallmatrix}\right]\left(\textstyle\int dv\right) \right)_{\left(\begin{smallmatrix}\varepsilon' \\ \varepsilon''\end{smallmatrix}\right)} \quad (\text{for } \mathrm{Prym}(\tilde{C}/C))$$

§ Schottky relation. (for $g=4$)

Recall $\forall\ g=1$ curve $\Rightarrow \vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4 = \vartheta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^4 + \vartheta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^4$

$$\begin{array}{lccc} & \tilde{C} & \xrightarrow{2:1} & C \\ \text{genus:} & 3 & & 2 \end{array}, \quad \eta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4 = \eta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^4 + \eta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^4 \ (\because g(\mathrm{Prym})=1 \Rightarrow \mathrm{Jac})$$

SJ Prop. $\Rightarrow$ (for $g=2$) $\vartheta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]^2 \vartheta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]^2 = \vartheta\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]^2 \vartheta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^2 + \vartheta\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]^2 \vartheta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]^2$

(always $\because A^2 = J(C)$ always).

Next $\begin{array}{ccc} \tilde{C} & \xrightarrow{2:1} & C \\ 5 & & 3 \end{array}$ switch $\vartheta \longrightarrow \eta$

SJ Prop $\Rightarrow$ ($g=3$) (also always holds)

$$\vartheta\left[\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right]\vartheta\left[\begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix}\right]\vartheta\left[\begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix}\right]\vartheta\left[\begin{smallmatrix}0&0&0\\1&1&0\end{smallmatrix}\right] = \vartheta[\ \ \ 1]\vartheta[1\ \ 1]\vartheta[\ 1\ 1]\vartheta[1\ 1\ 1] + \vartheta[\ \ \ 1]\vartheta[1\ \ 1]\vartheta[\ 1\ 1]\vartheta[1\ 1\ 1]$$

Next $\begin{array}{lccc} & \tilde{C} & \longrightarrow & C \\ g = & 7 & & 4 \end{array}$ switch θ to η

SJ Prop $\Rightarrow$ ($g=4$) $(\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta)^{1/2} = (\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta)^{1/2} + (\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta)^{1/2}$

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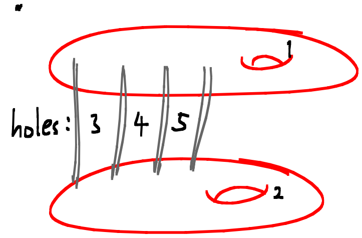
§ Nontrivial indeed !

$$\begin{array}{lcl} \text{involut}^{\underline{n}}:\ \imath \circlearrowright & C & \\ & {\scriptstyle 2:1}\downarrow h & \text{branch at} \\ g=1 & E & \ni p_1, \ldots\ldots, p_8 \end{array}$$



$g(C)=5$

$$\imath^*: H^{1,0}(C) \circlearrowleft \text{ s.t. } (\imath^*)^2 = 1 \Rightarrow \begin{array}{ccccc} H^{1,0}(C) & = & H^{1,0}(C)^+ & \oplus & H^{1,0}(C)^- \\ \text{dim: } \underset{g(C)}{5} & & \underset{g(E)}{1} & + & 4 \end{array}$$

$\Phi_K : C \longrightarrow \mathbb{P}^4 = |K_C| \ni q = \mathbb{P}(H^{1,0}(C)^+)$

Project from q :

$$\begin{tikzcd} C \arrow[r, hook, "\Phi_K"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}^4 \arrow[d, "\pi_q"] \\ E \arrow[r, hook, "f"', "\text{deg } 4"] & \mathbb{P}^3 \end{tikzcd}$$

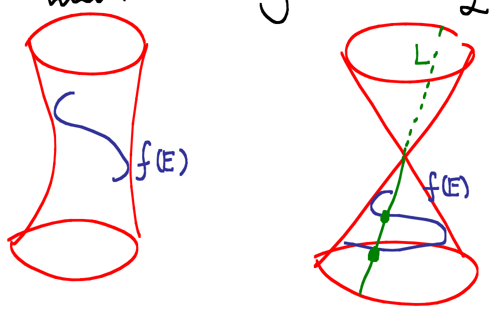
Claim. $f(E) =$ base locus of a pencil of quadrics

$$= \bigcap_{[\lambda_0,\lambda_1]\in\mathbb{P}^1} Q_{[\lambda_0,\lambda_1]} \subset \mathbb{P}^3$$

[Pf: $\{\text{quadric in } \mathbb{P}^3\} \simeq \mathbb{C}^{10}$ $\quad \binom{5}{3} = 10$

$\{\text{restric to } f(E)\} \simeq \mathbb{C}^8$ $\quad (\because \text{RR})$

$\exists$ 4 Singular quadric (cones) in Q's.



Q cone

$L \cap f(E) = p_1 + p_2$ 2-pts.

hyperplane section $= L + L'$

$$\Rightarrow \quad 2\,h^{-1}(p_1) + 2\,h^{-1}(p_2) \in |K_C|$$

i.e. $h^{-1}(p_1) + h^{-1}(p_2)$: theta char. of C.

Claim: Even theta char. of C.

Call them $L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)}$.

(reason: $\dim H^0(C, \mathcal{O}(h^{-1}(p_1) + h^{-1}(p_2))) \geq 2$ $\left(\because L \text{ moves in 1 dim}\right)$

If $\gneq 2 \Rightarrow C$ hyperelliptic

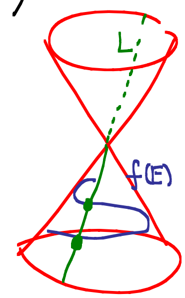
$\Rightarrow \Phi_K(C) \simeq \mathbb{P}^1$, but $f(E)$ $g=1$ $(\times)$

$L^{(j)}$ $\quad \dim H^0(C, L^{(j)}) = 2$

$\Rightarrow$ order 2 point $\begin{bmatrix}\delta(j)\\ \varepsilon(j)\end{bmatrix} \in J(C)$

w/ $\vartheta\begin{bmatrix}\delta(j)\\ \varepsilon(j)\end{bmatrix}(0;\Omega) = 0$ (Riemann sing. theorem.)

$\overset{\text{fact}}{\Longrightarrow}$ $\left.\begin{array}{l}\text{tangent cone to}\\ \vartheta\begin{bmatrix}\delta(j)\\ \varepsilon(j)\end{bmatrix}(u;\Omega) \text{ at } u=0\end{array}\right\} \rightsquigarrow$



!

(∃ exactly 4 even theta fu., vanish at $u=0$)

$\begin{bmatrix}\delta(j)\\ \varepsilon(j)\end{bmatrix} = \cancel{\begin{bmatrix}0\\0\end{bmatrix}}, \begin{bmatrix} & \\ 1 & \end{bmatrix}, \begin{bmatrix} & \\ & 1\end{bmatrix}, \begin{bmatrix} & \\ 1 & 1\end{bmatrix}$

Choose double cover of C s.t. SJ prop. $\rightsquigarrow$

$\underset{g=9}{\tilde{C}} \xrightarrow{2:1} \underset{5}{C}$ $\qquad \underbrace{\vartheta\begin{bmatrix}0 & \delta\\ 0 & \varepsilon\end{bmatrix}\vartheta\begin{bmatrix}0 & \delta\\ 1 & \varepsilon\end{bmatrix}}_{g=5} = \text{const.}\ \underbrace{\eta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}}_{g=4}$

$\Rightarrow$ $\underset{g=4}{\mathrm{Prym}(\tilde{C}/C)}$ s.t. $\eta\begin{bmatrix}0&0&0&0\\0&0&0&0\end{bmatrix} = 0 = \eta\begin{bmatrix}0&0&0&0\\0&0&0&1\end{bmatrix}$

"IF" Schottky relation holds for Prym. $\Rightarrow$

$\exists\ \eta\begin{bmatrix} & & & 1\\ j & k & \ell & \end{bmatrix} = 0$

SJ prop. $\Rightarrow$ 1 more quad. cone $Q \supseteq f(E)$ ($\cancel{\ast}$).

$\#$